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Rigidity of topological duals of spaces of formal series with respect to product topologies

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To Daniel Barsky, on the occasion of his retirement

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ABSTRACT

Even in spaces of formal power series is required a topology in order to legitimate some operations, in particular to compute infinite summations. In general the topologies considered are just a product of the topology of the base field, an inverse limit topology or a topology induced by a pseudo-valuation. As our main result we prove the following phenomenon: the (left and right) topological dual spaces of formal power series equipped with the product topology with respect to any Hausdorff division ring topology on the base division ring, are all the same, namely just the space of polynomials. As a consequence, this kind of rigidity forces linear maps, continuous with respect to any (and then to all) those topologies, to be defined by very particular infinite matrices similar to row-finite matrices.

1. Introduction

Some manipulations of formal power series require some topological properties in order to be legitimate. For instance, the usual substitution of a power series in one variable without constant term into another, or the existence of the star operation, related to the Möbius inversion formula, are usually treated using either an order function (a pseudo-valuation) or, equivalently (while more imprecise), arguing that only finitely many terms contribute to the calculation in each degree (this is the usual sort of arguments used

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in combinatorics[2]). In both cases is used, explicitly or not, a topology induced by a filtration: the "order" of some partial sums must increase indefinitely for the sum to be defined (and the operation to be legal). Quite naturally other topologies may be used: for instance if X is an infinite set, then the completion of the algebra $R\langle X\rangle$ of polynomials in non-commutative variables (where R is a commutative ring with a unit) with respect to the usual filtration (induced by the length of a word in the free monoid X^*) is the set of all series with only finitely many non-zero terms for each given length. The sum $\sum_{x\in X} x$ of the alphabet (the "zêta function" of X) does not even exist in this completion. In order to take such series into account – which could be fundamental for instance in the case of the Möbius inversion formula – we must use the product topology (with a discrete R) or consider a graduation of X so that there are only finitely many members of X in each degree. If some analytical investigations must be performed (such as convergence ray, resolution of differential equations, or some functional analysis), the discrete topology of $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is not sufficient anymore; the absolute value of $\mathbb{K}$ turns to be unavoidable. Other topologies may be used for particular needs.

Given a topology, compatible in a natural way with algebraic operations, on a space of formal power series with coefficients in a topological field (or division ring), it can be useful to consider continuous linear endomorphisms because they commute to infinite sums. Quite amazingly for a very large class of admissible topologies (namely product topologies with respect to Hausdorff topologies on the base field or division ring) it appears that these continuous linear maps may be seen as infinite matrices of a particular kind (each "row" is finitely supported) and that, independently of the topology chosen for the base field. In other terms, a linear map can be represented as some "row-finite" matrix if it is continuous for one and thus for all these topologies (see Section 7). Hence, in order to prove that an endomorphism is continuous with respect to some topology, it suffices to prove this property for the more convenient topology in the class. Note however that the representability of an endomorphism by a row-finite matrix is not sufficient to guarantee that it is continuous, because the representation is not faithful. Nevertheless each row-finite matrix represents a continuous linear endomorphism.

The explanation of this phenomenon relies on the following property of rigidity: the topological left and right duals of a given space of formal series, equipped with the product topology, with coefficients in some Hausdorff topological division ring, are forced to be the space of polynomials, independently of the topology on the base division ring, as soon as it is Hausdorff. This is the main result of the paper, presented in Section 3 (Theorem 5) and proved in Section 4. We also recast this result with a more category-theoretic flavor (in Section 6) in order to show that it provides a natural equivalence between two categories of vector spaces, extending some results of J. Dieudonné on linearly compact vector spaces [7]. Some direct consequences of this property of rigidity, in particular the representation by row-finite matrices, are presented in Section 7.

A formal power series is a set-theoretic map defined on a free monoid and with values in a ring, represented as an infinite sum. Hence, from a linear perspective (i.e., ignoring the multiplication of series, or in other words, replacing the free monoid by any set), and a bit radically, the theory of formal power series matches with that of spaces of ring-valued functions. Within this point of view, a polynomial turns to be a finitely-supported map. Furthermore, as soon as a topology is considered for both the base ring (for instance the discrete topology) and the function space (the product topology) these maps may also be represented as sums of summable families (see Remark 14 below), whence formal power series are recovered, and nothing is lost through such a general viewpoint. Therefore, if the multiplication of series is irrelevant for the desired

[2] One observes that the topological aspects of combinatorics are often neglected, and sometimes ignored even in the most famous textbooks. I cite Stanley [19, p. 196]: "Algebraically inclined readers can think of $\mathbb{K}\langle\langle X\rangle\rangle$ [Author's note: the space of formal power series in non-commutative variables in the set X over $\mathbb{K}$] as the completion of the monoid algebra of the free monoid X^* with respect to the ideal generated by X." but this is of course false as soon as X is infinite, because in this case the series $\sum_{x\in X} x$ belongs to $\mathbb{K}\langle\langle X\rangle\rangle$ while it does not belong to the completion of the monoidal algebra $\mathbb{K}[X^*]$ with respect to the $\mathfrak{M}$-adic topology induced by its augmentation ideal $\mathfrak{M} = X\mathbb{K}[X^*]$.

applications, like those presented hereafter, one can see formal series as functions, and this is precisely the way we deal with them in this contribution.

As a last word, one mentions that a part of the results of this contribution already appeared as Chapter 5 of the unpublished author's "habilitation" [16], and only in the commutative setting, i.e., for vector spaces over a field and not over a more general division ring.

2. Some notations and basic facts

This quite long section is devoted to the introduction of some notations, and definitions, and to recall some results which are useful for subsequent developments. Most of the notions occurring in this section are standard and may be found in many textbooks, but they are recalled for the reader convenience. In the first place, the algebraic notions of left, right and bimodules (and vector spaces, over division rings) are briefly exposed (Section 2.2), followed, in Section 2.3, by the same objects equipped with "compatible" topologies. Summability and its basic properties, which are fundamental for this paper, are presented in Section 2.4. Finally, in Section 2.5, are introduced the basic definitions concerning duality in the context of topological bimodules over (not necessarily commutative) topological rings; in particular one has to distinguish between left, right and two-sided (topological) duals.

2.1. Some general notations and definitions

Except in Section 6, the categories [14] do not play an important role in this contribution. Nonetheless the following helpful notation is introduced, and one freely uses a few very basic notions from category theory (such as functors, "forgetful" functors and sub-categories for instance). If $\mathbf{C}$ denotes a category, and a, b are objects of $\mathbf{C}$, then by $\mathbf{C}(a, b)$ is meant the set of all morphisms (or arrows or also maps) in $\mathbf{C}$ from a to b. For instance, let $\mathbf{Ab}$ be the category of abelian groups. Then, $\mathbf{Ab}(A, B)$ denotes the set of all group homomorphisms from A to B. Names for various categories are introduced in what follows.

The category $\mathbf{C}^{\mathsf{op}}$ is the *opposite category* of $\mathbf{C}$: both categories share the same objects, and each $f \in \mathbf{C}(a, b)$ is a member of $\mathbf{C}^{\mathsf{op}}(b, a)$. The composition of morphisms in $\mathbf{C}^{\mathsf{op}}$ is the opposite of that in $\mathbf{C}$. This notion is important to define some dual functors (see Section 2.5).

2.2. (Bi)modules and (bi-)vector spaces

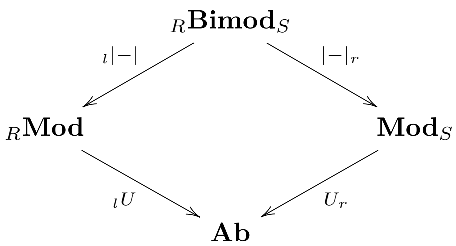
Throughout this contribution, unless stated the contrary, by a *ring* R is meant a *(not necessarily commutative) ring with an identity element* (i.e., a unital ring), which is denoted by 1_R (its zero is denoted by 0_R). A *division ring* $\mathbb{K}$ (see [10]) is a non-zero ring (i.e., $0_{\mathbb{K}} \neq 1_{\mathbb{K}}$) whose non-zero members are invertible. A commutative division ring thus is a field.

An abelian group M is said to be a *left R-module* (also called a *left module over R*) if it is equipped with a *left R-action*, i.e., a map $(r, v) \in R \times M \mapsto rv \in M$ such that $1_R v = v$ (i.e., each module is unital), and $(rs)v = r(sv)$, $r(v + w) = rv + rw$ for each $r, s \in R$ and each $v, w \in M$. Symmetrically one may introduce a notion of *right R-module* (the *right R-action* is then denoted by $(v, r) \in M \times R \to vr \in M$). Let S be another ring (with a unit). An abelian group M which is both a left R-module and a right S-module is called a R-S-bimodule whenever the left R-action and the right S-action are compatible, i.e., $r(vs) = (rv)s$, $r \in R$, $v \in V$, $s \in S$; the common value of $r(vs)$ and of $(rv)s$ is denoted by rvs. When R and S are both division rings, then one obtains *left* (or *right*) *R-vector spaces*, and *R-S-bi-vector spaces*.

Remark 1. If R is a commutative ring, then the structures of left and of right modules over R coincide, so that one just talks about *R-modules*. Any R-module is a R-R-bimodule with the left action as right action (or vice versa). However it is easy to observe that not all R-R-bimodules arise in this way. Likewise when R is a field, any $\mathbb{K}$-vector space is a $\mathbb{K}$-$\mathbb{K}$-bi-vector space.

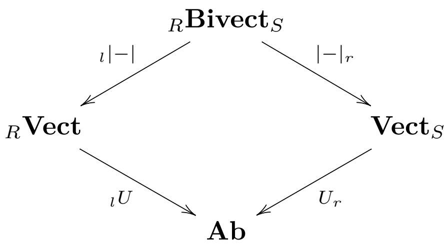
Given two left (or right) R-modules (or $\mathbb{K}$-vector spaces) M, N, and a group homomorphism ϕ from M to N, ϕ is said to be a *left* (or *right*) *R-linear map* if $\phi(rv) = r\phi(v)$ (or $\phi(vr) = \phi(v)r$), $r \in R$, $v \in V$. If M and N both are R-S-bimodules (or R-S-bi-vector spaces, in case both R and S are division rings), and if $\phi\colon M \to N$ is a group homomorphism, then ϕ is a *R-S-linear map*, or a *left R-linear and right S-linear map*, when $\phi(rvs) = r\phi(v)s$, $r \in R$, $s \in S$, $v \in V$.

One introduces the categories ${}_R\mathbf{Mod}$ (respectively, ${}_R\mathbf{Vect}$, when R is a division ring) of left R-modules (respectively, left R-vector spaces) with left R-linear maps, $\mathbf{Mod}_S$ (respectively, $\mathbf{Vect}_S$) of right S-modules (respectively, right S-vector spaces, when S is a division ring) with right S-linear maps, ${}_R\mathbf{Bimod}_S$ (respectively, ${}_R\mathbf{Bivect}_S$) of R-S-bimodules (respectively, R-S-bi-vector spaces, when R, S are division rings) with R-S-linear maps. One has the following commutative diagrams of obvious forgetful functors.



(1)

and also



(2)

The common value ${}_lU \circ {}_l|-| = U_r \circ |-|_r$ in both diagrams is sometimes denoted by $|-|$. In details, by ${}_l|-|$ (respectively, $|-|_r$) is meant the functor that consists in forgetting the structure of right (respectively, left) R-module (or R-vector space) of a R-S-bimodule (respectively, R-S-bi-vector space), i.e., ${}_l|M|$ (respectively, $|M|_r$) is the *underlying left R-module* (respectively, *underlying right S-module*) of a R-S-bimodule M, while ${}_lU$ (respectively, U_r) is the functor that consists in forgetting the left (respectively, right) R-action (respectively S-action) of a left R-module, or left R-vector space (respectively, right S-module, or right S-vector space). Finally, $|-|$ forgets both module (or vector space) structures on a bimodule (or bi-vector space) and provides the *underlying abelian group* of a bimodule.

Remark 2. If R is a ring (or a division ring), then R itself turns to be a left (respectively, right) R-module (or vector space), denoted by ${}_lR$ (respectively, R_r) with left (respectively, right) R-action given by left (respectively, right) multiplication. Actually, R, with the above actions, acquires a structure of a R-R-bimodule (or R-R-bi-vector space), denoted by ${}_lR_r$. Of course one has ${}_l|{}_lR_r| = {}_lR$, $|{}_lR_r|_r = R_r$, and $|{}_lR_r|$ is the abelian group structure of R.

2.3. Topologies compatible with module (vector space) structures

2.3.1. Initial and product topologies

The following notions may be found, e.g., in [3].

Let E be a set, and let $\mathcal{F}$ be a collection of set-theoretic maps, with common domain E, $\phi: E \to E_\phi$, where each (E_ϕ, τ_ϕ) is a topological space. The *initial topology* induced by $\mathcal{F}$ on E is the coarsest topology that makes continuous each ϕ. This topology is Hausdorff if, and only if, $\mathcal{F}$ *separates* the members of E (i.e., for every $x \neq y$ in E, there exists some $\phi \in \mathcal{F}$ such that $\phi(x) \neq \phi(y)$), and (E_ϕ, τ_ϕ) is Hausdorff for each $\phi \in \mathcal{F}$.

Especially, if (F_i, τ_i) is a topological space for each i in some index set I, then one can equip the cartesian product $\prod_{i\in I} F_i$ with the initial topology induced by the projections $\pi_j: \prod_{i\in I} F_i \to F_j$, $j \in I$, which is called the *product topology*. Of course, it is Hausdorff as soon as each (F_i, τ_i) is a Hausdorff space.

As a particular instance, if E is the function space F^X, where X is a set and (F, τ) is a topological space, then the *product topology* (or *topology of simple convergence* or *function topology*) on F^X is the initial topology, denoted by $\sqcap^\tau_X$, induced by the canonical projections $\pi_x: f \mapsto f(x)$, $x \in X$, from E onto F. It is characterized by the following universal property. For every topological space (Y, τ_Y), the map $f: Y \to F^X$ is continuous if, and only if, every map $\pi_x \circ f: Y \to F$, $x \in X$, is continuous. Once again it is a Hausdorff space whenever (F, τ) is Hausdorff.

2.3.2. Topological (bi)modules ((bi-)vector spaces)

Let R be a (not necessarily commutative) ring with a unit 1_R (and zero 0_R). It is said to be a *topological ring* when it is equipped with a topology τ (not necessarily Hausdorff) with respect to which the ring operations, $x \mapsto -x$, $(x, y) \mapsto x + y$, $(x, y) \mapsto xy$, are continuous, when is considered on $R \times R$ the product topology defined by τ on each factor. One may also define the notion of a *topological group* in an obvious way.

A division ring (respectively, field) $\mathbb{K}$, which is also a topological ring, is said to be a *topological division ring* (respectively, *topological field*) when the map $x \mapsto x^{-1}$ is continuous on $\mathbb{K}^* = \mathbb{K} \setminus \{0_\mathbb{K}\}$ (equipped with the sub-space topology), i.e., the group of units $\mathbb{K}^*$ is a topological group under the sub-space topology.

Remark 3. Any ring (respectively, division ring or group) is naturally a topological ring (respectively, division ring or group) when equipped with the discrete topology d or the trivial topology t. In the later case, it is of course not Hausdorff (unless for the zero ring and the zero group).

If (R, τ) (respectively, $(\mathbb{K}, \tau)$) is a topological ring (respectively, a topological division ring), then any left (or right) R-module M (respectively, left (or right) $\mathbb{K}$-vector space) is said to be a left (or right) *topological* (R, τ)-*module* (respectively, a left (or right) *topological* $(\mathbb{K}, \tau)$-*vector space*) if it is equipped with a topology (Hausdorff or not) that makes continuous the module maps $v \mapsto -v$, $(v, w) \mapsto v + w$, $(r, v) \mapsto rv$ (again $M \times M$ and $R \times M$ are considered as topological spaces under the product topology). If, (S, μ) is also a topological ring, and if (M, σ) is a R-S-bimodule which is both a topological left R-module and a topological right S-module, then (M, σ) is said to be a *topological* (R, τ)-(S, μ)-*bimodule*. Of course, one similarly gets a *topological* (R, τ)-(S, μ)-*bi-vector space* when (R, τ) and (S, μ) are topological division rings.

Remark 4. Observe that $({}_lR_r, \tau)$ (respectively, $({}_lR, \tau)$, (R_r, τ), $({}_l|R|_r, \tau)$) is a topological (R, τ)-(R, τ)-bimodule (respectively, topological left (R, τ)-module, topological right (R, τ)-module, topological group), and we get the corresponding results for vector spaces when (R, τ) is a topological division ring.

Remark 5. When (R, τ) is a commutative topological ring (respectively, a topological field) and when (M, σ) is a topological (R, τ)-module (respectively, a topological (R, τ)-vector space), then it is, in a natural way (see Remark 1), a topological (R, τ)-(R, τ)-bimodule (respectively, a topological (R, τ)-(R, τ)-bi-vector space).

Let (R,τ) and (S,μ) be topological rings (or division rings), and let M be a topological left (or right) R-module (or vector space), or a R-S-bimodule (or R-S-bi-vector space). Let $\mathcal{F}$ be a set of maps such that for each $\phi \in \mathcal{F}$, $\phi\colon M \to M_\phi$ is a left R-linear (or right R-linear or R-S-linear) map, and (M_ϕ, τ_ϕ) is a topological left (or right) (R,τ)-module (or vector space) or an (R,τ)-(S,μ)-bimodule (or bi-vector space), i.e., with the same structure as that on M, then the initial topology induced by $\mathcal{F}$ on M turns M into a topological left (or right) (R,τ)-module (or vector space) or a topological (R,τ)-(S,μ)-bimodule (or bi-vector space). Likewise an initial topology induced by a set of maps of the form $\phi\colon G \to G_\phi$, where each (G_ϕ, τ_ϕ) is a topological group, endows a group G with a structure of a topological group.

Remark 6. In particular if $(M_i)_{i\in I}$ is a family of topological groups or (bi)modules or (bi-)vector spaces, then $\prod_{i\in I} M_i$, with the product topology, turns to be a topological group or (bi)module or (bi-)vector space. Hence, as soon as each M_i is Hausdorff, this product topology also is Hausdorff. Moreover, if M_i is a topological (R,τ)-(S,μ)-bimodule (or bi-vector space) for each $i \in I$, then its underlying left R-module structure ${}_l|\prod_{i\in I} M_i| = \prod_{i\in I} {}_l|M_i|$, its underlying right S-module (or vector space) structure $|\prod_{i\in I} M_i|_r = \prod_{i\in I} |M_i|_r$, and its underlying abelian group structure $|\prod_{i\in I} M_i| = \prod_{i\in I} |M_i|$, are topological modules (or vector spaces) and topological groups (with the product topology).

A topology compatible with a ring (division ring, field, group, module, vector space) structure as above is also referred to as a *ring* (or *division ring, field, group, module, vector space*) *topology*.

More details about the above notions may be found in [21].

2.4. Summability

Many intermediary results of this paper require the notion, and some properties, of a summable family in a topological ring (or module). We recall them without any proof; we freely used them in the sequel, and refer to [21] for further information concerning this concept.

Definition 1. Let G be a Hausdorff abelian group (that is, an abelian group – in additive notation – with a Hausdorff topology such that the group operations, addition and inversion, are continuous), and let $(x_i)_{i\in I}$ be a family of elements of G. An element $s \in G$ is the *sum* of the *summable family* $(x_i)_{i\in I}$ if, and only if, for each neighborhood V of s there exists a finite subset $J \subseteq I$ such that $\sum_{j\in J} x_j \in V$.

The sum s of a summable family $(x_i)_{i\in I}$ of elements of G is usually denoted by $\sum_{i\in I} x_i$ (of course it is unique, when it exists, due to the separation axiom).

Proposition 1. *If $(x_i)_{i\in I}$ is a summable family of elements of a Hausdorff abelian group G having a sum s, then for any permutation σ of I, s is also the sum of the summable family $(x_{\sigma(i)})_{i\in I}$.*

Proposition 2. *If $(x_i)_{i\in I}$ is a summable family of elements of a Hausdorff abelian group G, then for every neighborhood V of zero, $x_i \in V$ for all but finitely many $i \in I$.*

Proposition 3. *Let G be the cartesian product of a family $(G_\lambda)_{\lambda\in L}$ of Hausdorff abelian groups (G has the product topology). Then s is the sum of a family $(x_i)_{i\in I}$ of elements of G if, and only if, $\pi_\lambda(s)$ is the sum of $(\pi_\lambda(x_i))_{i\in I}$ for each $\lambda \in L$ (where π_λ is the canonical projection from G onto G_λ).*

Proposition 4. *If ϕ is a continuous homomorphism from a Hausdorff abelian group G to a Hausdorff abelian group H, and if $(x_i)_{i\in I}$ is a summable family of elements of G, then $(\phi(x_i))_{i\in I}$ is summable in H, and* $\sum_{i\in I}\phi(x_i) = \phi\left(\sum_{i\in I} x_i\right)$.

Remark 7. The above propositions may be applied when G is the underlying abelian group of a Hausdorff topological (bi, left or right) module (or vector space) according to Remark 4 (in particular, Proposition 3, when the module into consideration has a product topology, and is Hausdorff, see Remark 6, and Proposition 4, with ϕ a continuous linear map between Hausdorff topological modules, since it is also a continuous group homomorphism between their underlying topological abelian groups).

2.5. Duality

One recalls some well-known notions about (algebraic and topological) dual spaces in a non-commutative setting (see [7] for instance).

2.5.1. Algebraic duals

Let R, S, T be unital rings.

Let M be a R-S-bimodule and N be a R-T-bimodule. The abelian group (under point-wise operations) ${}_R\mathbf{Mod}({}_l|M|, {}_l|N|)$ (or ${}_R\mathbf{Vect}({}_l|M|, {}_l|N|)$ when R is a division ring) of all left R-linear maps (i.e., $\phi(rx) = r\phi(x)$) is an S-T-bimodule with left S-action $(s \cdot \phi)(x) = \phi(xs)$ and right T-action $(\phi \cdot t)(x) = \phi(x)t$. In particular when $R = S = T$, then it is a R-R-bimodule (or a R-R-bi-vector space, in case R is a division ring), denoted by ${}^*M$ and called the *left (algebraic) dual of* M when $N = {}_lR_r$.

Let M be an S-R-bimodule and N be a T-R-bimodule. The abelian group (under point-wise operations) $\mathbf{Mod}_R(|M|_r, |N|_r)$ (or $\mathbf{Vect}_R(|M|_r, |N|_r)$ if R is a division ring) of all right R-linear maps (i.e., $\phi(xr) = \phi(x)r$) is a T-S-bimodule with left T-action $(t \cdot \phi)(x) = t\phi(x)$ and right S-action $(\phi \cdot s)(x) = \phi(sx)$. With $R = S = T$ one gets a R-R-bimodule (or a R-R-bi-vector space for R a division ring), denoted by M^* and called the *right (algebraic) dual of* M when $N = {}_lR_r$.

Let M and N be S-T-bimodules. The set ${}_S\mathbf{Bimod}_T(M, N)$ (or the set ${}_S\mathbf{Bivect}_T(M, N)$ when both S and T are division rings) is an abelian group (under point-wise operations). When $S = R = T$ and $N = {}_lR_r$, it is called the *two-sided (algebraic) dual* ${}^*M^*$ of M. Of course, ${}^*M^* = {}^*M \cap M^*$.

Remark 8. Observe that ${}^*M^*$ is of course not the right dual of the left dual $({}^*M)^*$ of M nor the left dual of the right dual ${}^*(M^*)$ of M.

With the above different duals, one gets three functors, namely

$$
\begin{aligned}
{}^*(-)\colon {}_R\mathbf{Bimod}_R^{\mathsf{op}} &\to {}_R\mathbf{Bimod}_R,\\
(-)^*\colon {}_R\mathbf{Bimod}_R^{\mathsf{op}} &\to {}_R\mathbf{Bimod}_R
\end{aligned}
$$

and

$$
{}^*(-)^*\colon {}_R\mathbf{Bimod}_R^{\mathsf{op}} \to \mathbf{Ab}.
$$

(For instance the first functor sends a left and right R-linear map $\phi\colon N \to M$ to the left and right R-linear map ${}^*\phi\colon \ell \in {}^*M \mapsto \ell \circ \phi \in {}^*N$.) Similarly, assuming that R is a division ring, one gets

$$
\begin{aligned}
{}^*(-)\colon {}_R\mathbf{Bivect}_R^{\mathsf{op}} &\to {}_R\mathbf{Bivect}_R,\\
(-)^*\colon {}_R\mathbf{Bivect}_R^{\mathsf{op}} &\to {}_R\mathbf{Bivect}_R
\end{aligned}
$$

and

$$^*(-)^*\colon {}_R\mathbf{Bivect}_R^{\mathsf{op}} \to \mathbf{Ab}.$$

Remark 9. If R is a commutative ring, then it is possible to see the category ${}_R\mathbf{Mod}$ as a sub-category of ${}_R\mathbf{Bimod}_R$ where the right action is taken as being the same as the left action (and similarly, if $\mathbb{K}$ is a field, ${}_\mathbb{K}\mathbf{Vect}$ is a sub-category of ${}_\mathbb{K}\mathbf{Bivect}_\mathbb{K}$) as explained in Remark 1. In such a situation the three above dual functors collapse into a unique one $(-)^*\colon {}_R\mathbf{Mod}^{\mathsf{op}} \to {}_R\mathbf{Mod}$ (and similarly, $(-)^*\colon {}_\mathbb{K}\mathbf{Vect}^{\mathsf{op}} \to {}_\mathbb{K}\mathbf{Vect}$).

2.5.2. Topological duals

For a pair of topological spaces (E_i, τ_i), $i = 1, 2$, let $\mathbf{Top}((E_1, \tau_1), (E_2, \tau_2))$ be the set of all continuous maps from (E_1, τ_1) to (E_2, τ_2).

Given any topological bimodule (M, σ) over a topological ring (R, τ), one defines its *left topological dual* ${}'(M, \sigma) = {}^*M \cap \mathbf{Top}((M, \sigma), (R, r))$, its *right topological dual* $(M, \sigma)' = M^* \cap \mathbf{Top}((M, \sigma), (R, r))$ and its *two-sided topological dual* ${}'(M, \sigma)' = {}^*M^* \cap \mathbf{Top}((M, \sigma), (R, \tau)) = {}'(M, \sigma) \cap (M, \sigma)'$.

It is quite immediate that ${}'(M, \sigma)$ (respectively, $(M, \sigma)'$) is a sub-R-R-bimodule of ${}^*M$ (respectively, M^*), and ${}'(M, \sigma)'$ is a sub-group of ${}^*M^*$. Denoting by ${}_{(R,\tau)}\mathbf{TopBimod}_{(R,\tau)}$ the category of all topological (R, τ)-bimodules (with continuous left and right linear maps), one gets functors

$$\begin{aligned} {}'(-)\colon {}_{(R,\tau)}\mathbf{TopBimod}_{(R,\tau)}^{\mathsf{op}} &\to {}_R\mathbf{Bimod}_R, \\ (-)'\colon {}_{(R,\tau)}\mathbf{TopBimod}_{(R,\tau)}^{\mathsf{op}} &\to {}_R\mathbf{Bimod}_R \end{aligned}$$

and

$${}'(-)'\colon {}_{(R,\tau)}\mathbf{TopBimod}_{(R,\tau)}^{\mathsf{op}} \to \mathbf{Ab}.$$

Of course, when $(\mathbb{K}, \tau)$ is a topological division ring, then one denotes by ${}_{(\mathbb{K},\tau)}\mathbf{TopBivect}_{(\mathbb{K},\tau)}$ the category of all topological $(\mathbb{K}, \tau)$-vector spaces (with continuous left and right linear maps). In this situation, one has the following functors

$$\begin{aligned} {}'(-)\colon {}_{(\mathbb{K},\tau)}\mathbf{TopBivect}_{(\mathbb{K},\tau)}^{\mathsf{op}} &\to {}_\mathbb{K}\mathbf{Bivect}_\mathbb{K}, \\ (-)'\colon {}_{(\mathbb{K},\tau)}\mathbf{TopBivect}_{(\mathbb{K},\tau)}^{\mathsf{op}} &\to {}_\mathbb{K}\mathbf{Bivect}_\mathbb{K} \end{aligned}$$

and

$${}'(-)'\colon {}_{(\mathbb{K},\tau)}\mathbf{TopBivect}_{(\mathbb{K},\tau)}^{\mathsf{op}} \to \mathbf{Ab}.$$

Remark 10. When (R, τ) is a commutative topological ring, then the category ${}_{(R,\tau)}\mathbf{TopMod}$ of all topological (R, τ)-left modules (with continuous left linear maps) may be identified with a sub-category of ${}_{(R,\tau)}\mathbf{TopBimod}_{(R,\tau)}$, and similarly when $(\mathbb{K}, \tau)$ is a topological field, then the category ${}_{(\mathbb{K},\tau)}\mathbf{TopVect}$ of all topological $(\mathbb{K}, \tau)$-vector spaces is a sub-category of ${}_{(\mathbb{K},\tau)}\mathbf{TopBivect}_{(\mathbb{K},\tau)}$; see Remark 5. In this situation, there is only one topological dual functor of interest: $(-)^*\colon {}_{(R,\tau)}\mathbf{TopMod}^{\mathsf{op}} \to {}_R\mathbf{Mod}$ (and $(-)^*\colon {}_{(\mathbb{K},\tau)}\mathbf{TopVect}^{\mathsf{op}} \to {}_\mathbb{K}\mathbf{Vect}$).

3. Statement of the main result

When X is a sets, then for each map $f \in R^X$, one defines its *support* $\mathsf{supp}(f) = \{ x \in X\colon f(x) \neq 0 \}$. Such a map f is said to be a *finitely supported map*, or a map with a *finite support*, when $\mathsf{supp}(f)$ is finite.

The set of all finitely supported maps is denoted by $R^{(X)}$. The set R^X is actually an abelian group under point-wise addition. This abelian group also admits a left R-action $(r, f) \mapsto (x \mapsto rf(x))$ and a right R-action $(f, r) \mapsto (x \mapsto f(x)r)$, and this turns R^X into a R-R-bimodule (or bi-vector space, when R is a division ring), which is denoted by ${}_lR_r^X$ (see also Remark 2). Its underlying left (respectively, right) R-module (or vector space) is denoted by ${}_lR^X$ (respectively, R_r^X). It is quite clear that $R^{(X)}$, with the same actions as above, may be seen either as a R-R-bimodule (or bi-vector space) ${}_lR_r^{(X)}$, a left R-module (or vector space) ${}_lR^{(X)}$ or a right R-module (or vector space) $R_r^{(X)}$.

Observe that if (R, τ) (respectively, $(\mathbb{K}, \tau)$) is a topological ring (respectively, topological division ring), and X is any set, then ${}_lR_r^X$ (respectively, ${}_l\mathbb{K}_r^X$) is a topological (R, τ)-bimodule (respectively, topological $(\mathbb{K}, \tau)$-bi-vector space) when endowed with the product topology $\sqcap_X^\tau$ (see Section 2.3.1 and Remark 6). As soon as τ is Hausdorff, $({}_lR_r^X, \sqcap_X^\tau)$ (respectively, $({}_l\mathbb{K}_r^X, \sqcap_X^\tau)$) also is.

For every $x_0 \in X$, we define the *characteristic function* (or *Dirac mass*)

$$\begin{array}{rccl} \delta_{x_0}\colon & X & \to & R \\ & x & \mapsto & \begin{cases} 1_R & \text{if } x = x_0, \\ 0_R & \text{otherwise.} \end{cases} \end{array} \tag{3}$$

The left (respectively, right) R-module ${}_lR^{(X)}$ (respectively, $R_r^{(X)}$) is free with left (respectively, right) basis $\{\, \delta_x : x \in X \,\}$.

Remark 11. In general, the R-R-bimodule ${}_lR_r^{(X)}$, even if it is free as a left or a right R-module, with basis $\{\, \delta_x : x \in X \,\}$, is not free as R-R-bimodule (because it satisfies the relation $r\delta_x = \delta_x r$, for each $r \in R$, $x \in X$, which is not a consequence of the axioms of bimodules; in Section 6 is studied this particular kind of bimodules, under the name of *double modules*).

Let us introduce the *left evaluation map*

$$\begin{array}{rlcl} \Gamma\colon & {}_lR_r^{(X)} \otimes_R {}_lR_r^X & \to & R \\ & p \otimes f & \mapsto & \displaystyle\sum_{x \in X} p(x) f(x). \end{array} \tag{4}$$

As usually it is treated as an R-bilinear map ${}_l\langle \cdot, \cdot \rangle$, called *dual pairing* (or *canonical bilinear form*, see [4] or [12]), *i.e.*,

$$\Gamma(p \otimes f) = {}_l\langle p, f \rangle.$$

In particular, for every $x \in X$, $\Gamma(\delta_x \otimes f) = \langle \delta_x, f \rangle = f(x)$, and then $\pi_x : f \mapsto \langle \delta_x, f \rangle = f(x)$ is the *projection* of R^X onto $R\delta_x \cong R$. The dual pairing has the obvious properties of non-degeneracy:

1. for every $p \in R^{(X)} \setminus \{0\}$, there is some $f \in R^X$ such that ${}_l\langle p, f \rangle \neq 0$,
2. for every $f \in R^X \setminus \{0\}$, there is some $p \in R^{(X)}$ such that ${}_l\langle p, f \rangle \neq 0$.

By symmetry one also defines the *right evaluation map*

$$\begin{array}{rlcl} \Delta\colon & {}_lR_r^X \otimes_R {}_lR_r^{(X)} & \to & R \\ & f \otimes p & \mapsto & \displaystyle\sum_{x \in X} f(x) p(x) \end{array} \tag{5}$$

which is also bilinear and non-degenerate (more precisely, the associated R-bilinear map $\langle \cdot, \cdot \rangle_r : {}_lR_r^X \times {}_lR_r^{(X)} \to R$, defined by $\langle f, p \rangle_r = \Delta(f \otimes p)$, is non-degenerate).

At last, let $Z(R)$ be the *center* of the ring R, i.e., is the subset of R consisting of all those elements s of R such that $sr = rs$ for all $r \in R$ (the center is a commutative sub-ring of R). Let us introduce the following group homomorphism, called the *two-sided evaluation map*

$$\begin{array}{llll} \Xi\colon & Z(R)^{(X)} \otimes_{\mathbb{Z}} R^X & \to & R \\ & p \otimes f & \mapsto & \sum_{x \in X} p(x) f(x). \end{array} \tag{6}$$

By definition, for each $p \in Z(R)^{(X)}$ and each $f \in R^X$, ${}_l\langle p, f\rangle = \Xi(p \otimes f) = \langle f, p\rangle_r$. Note that its associated bi-additive[3] map $(p, f) \mapsto \Xi(p \otimes f)$ is not R-bilinear (because $Z(R)$ is generally not a right (nor a left) R-module).

Remark 12. When R is a commutative ring with a unit, then $\Gamma = \Delta \circ \sigma$, where $\sigma(f \otimes p) = p \otimes f$ is the usual twist. Moreover, because in this case $Z(R) = R$, the following diagram commutes in the category of abelian groups (where *can* denotes the canonical epimorphism).

$$\begin{array}{ccc} R^{(X)} \otimes_{\mathbb{Z}} R^X & \xrightarrow{\ can\ } & R^{(X)} \otimes_R R^X \\ \Xi \downarrow & \overset{\Gamma}{\swarrow} & \downarrow \sigma \\ R & \xleftarrow[\ \Delta\ ]{} & R^X \otimes_R R^{(X)} \end{array} \tag{7}$$

The objective of this paper is to prove the following "rigidity" result, and to explore some of its main consequences.

Theorem 5. *Let $(\mathbb{K}, \tau)$ be a Hausdorff topological division ring, and let X be any set. Then, the right topological dual $({}_l\mathbb{K}_r^X, \sqcap_X^\tau)'$ and the left topological dual ${}^\backprime({}_l\mathbb{K}_r^X, \sqcap_X^\tau)$ of $({}_l\mathbb{K}_r^X, \sqcap_X^\tau)$ are isomorphic (as $\mathbb{K}$-$\mathbb{K}$-bi-vector spaces) to ${}_l\mathbb{K}_r^{(X)}$. The two-sided topological dual ${}^\backprime({}_l\mathbb{K}_r^X, \sqcap_X^\tau)'$ of $({}_l\mathbb{K}_r^X, \sqcap_X^\tau)$ is isomorphic to $Z(\mathbb{K})^{(X)}$ (as abelian groups).*

Remark 13. Of course, when $\mathbb{K}$ is a field, then Theorem 5 amounts to say that $(\mathbb{K}^X, \sqcap_X^\tau)' \simeq \mathbb{K}^{(X)}$ as vector spaces.

In order to illustrate the scope of this result, let us assume for awhile that $(\mathbb{K}, \mathsf{d})$ is a discrete field, and that (V, τ) is a Hausdorff topological $(\mathbb{K}, \mathsf{d})$-vector space. The topology τ on V is said to be *linear* if, and only if, it has a neighborhood basis of zero consisting of sub-vector spaces.

Proposition 6. *([7,13]) Let V be a $\mathbb{K}$-vector space together with a linear topology τ. Then the following conditions are equivalent:*

1. *(V, τ) is complete, and all its open subspaces are of finite codimension.*
2. *V is an inverse limit of discrete finite-dimensional vector spaces, with the inverse limit topology.*
3. *V is isomorphic, as a topological vector space, to the algebraic dual W^* of a discrete vector space W, with the topology of simple convergence. Equivalently, (V, τ) is isomorphic to $(\mathbb{K}^X, \sqcap_X^{\mathsf{d}})$, with the product topology, for some set X.*

[3] Given abelian groups A, B, C, a map $\phi\colon A \times B \to C$ is said to be *bi-additive* when it is a homomorphism of groups in each variable.

A topological vector space with the above equivalent properties is said to be *linearly compact*. As a minor consequence of our main result (Theorem 5) we obtain a characterization of the topological duals of these linearly compact spaces: the topological dual of any linearly compact vector space is isomorphic to some $\mathbb{K}^{(X)}$. This actually establishes an equivalence between the opposite of the category of $\mathbb{K}$-vector spaces and the category of linearly compact $\mathbb{K}$-vector spaces (see [1, Proposition 24.8, p. 107]). This result is extended in Section 6.

4. The proof of Theorem 5

Lemma 7. *Let R be a (not necessarily commutative) ring with unit, and let X be a set. Let us define*

$$\begin{array}{llll} \lambda\colon & R^{(X)} & \to & R^{R^X} \\ & p & \mapsto & \left(\begin{array}{llll} \lambda(p)\colon & R^X & \to & R \\ & f & \mapsto & {}_{\iota}\langle p, f\rangle \end{array}\right). \end{array} \tag{8}$$

Then, for every $p \in R^{(X)}$, $\lambda(p) \in ({}_{\iota}R_r^X)^*$, λ is a left and right R-linear map from ${}_{\iota}R_r^{(X)}$ to $({}_{\iota}R_r^X)^*$, and is one-to-one.

Proof. The first property is obvious (because it is equivalent to ${}_{\iota}\langle p, fr\rangle = {}_{\iota}\langle p, f\rangle r$, $p \in R^{(X)}$, $f \in R^X$ and $r \in R$). Let us check that λ commutes to the left and right R-actions. Let $p \in R^{(X)}$, $r \in R$. Then, for all $f \in R^X$, $\lambda(rp)(f) = r\lambda(p)(f) = (r \cdot \lambda(p))(f)$ and $\lambda(pr)(f) = {}_{\iota}\langle pr, f\rangle = {}_{\iota}\langle p, rf\rangle = (\lambda(p))(rf) = (\lambda(p) \cdot r)(f)$. It remains to see that λ is one-to-one. Let $p \in R^{(X)}$ such that $\lambda(p) = 0$, then for every $f \in R^X$, $\lambda(p)(f) = 0$, and in particular, for every $x \in X$, $0 = \lambda(p)(\delta_x) = {}_{\iota}\langle p, \delta_x\rangle = p(x)$, in such a way that $p = 0$. □

Symmetrically (and without further ado), one gets the following.

Lemma 8. *Let R be a (not necessarily commutative) ring with unit, and let X be a set. Let us define*

$$\begin{array}{llll} \rho\colon & R^{(X)} & \to & R^{R^X} \\ & p & \mapsto & \left(\begin{array}{llll} \rho(p)\colon & R^X & \to & R \\ & f & \mapsto & \langle f, p\rangle_r \end{array}\right). \end{array} \tag{9}$$

Then, for every $p \in R^{(X)}$, $\rho(p) \in {}^*({}_{\iota}R_r^X)$, ρ is a left and right R-linear map from ${}_{\iota}R_r^{(X)}$ to ${}^*({}_{\iota}R_r^X)$, and is one-to-one.

In a similar way, the following holds.

Lemma 9. *Let R be a (not necessarily commutative) ring with unit, and let X be a set. Let us define*

$$\begin{array}{llll} \zeta\colon & Z(R)^{(X)} & \to & R^{R^X} \\ & p & \mapsto & \left(\begin{array}{llll} \zeta(p)\colon & R^X & \to & R \\ & f & \mapsto & {}_{\iota}\langle p, f\rangle = \langle f, p\rangle_r \end{array}\right). \end{array} \tag{10}$$

Then, for every $p \in Z(R)^{(X)}$, $\zeta(p) \in {}^*({}_{\iota}R_r^X)^*$, ζ is a group homomorphism from $Z(R)^{(X)}$ to ${}^*({}_{\iota}R_r^X)^*$, and is one-to-one.

Proof. The first point follows from the fact that $p(x) \in Z(R)$ for each $x \in X$. The second point is obvious, and the last point is proved as in the proof of Lemma 7. □

Lemma 10. *Let us assume that (R,τ) is a topological ring (Hausdorff or not) with a unit, and that ${}_lR_r^X$ has the product topology. Then for every $p \in R^{(X)}$, $\lambda(p)$ is continuous, i.e., $\lambda(p) \in ({}_lR_r^X, \sqcap_X^\tau)'$. Whence, by Lemma 7, $\lambda \in {}_R\mathbf{Bimod}_R({}_lR_r^{(X)}, ({}_lR_r^X, \sqcap_X^\tau)')$.*

Proof. It is clear since $\lambda(p)$ is a finite sum of (scalar multiples) of projections, $\lambda(p) = \sum_{x\in X} p(x)_l\langle\delta_x, \cdot\rangle$ (sum with finitely many non-zero terms). □

Again by symmetry, the following result is obtained.

Lemma 11. *Let us assume that (R,τ) is a topological ring (Hausdorff or not) with a unit, and that ${}_lR_r^X$ has the product topology. Then for every $p \in R^{(X)}$, $\rho(p)$ is continuous, i.e., $\rho(p) \in {}'({}_lR_r^X, \sqcap_X^\tau)$. Whence, by Lemma 8, $\rho \in {}_R\mathbf{Bimod}_R({}_lR_r^{(X)}, {}'({}_lR_r^X, \sqcap_X^\tau))$.*

The "two-sided" corresponding result is given below.

Lemma 12. *Let us assume that (R,τ) is a topological ring (Hausdorff or not) with a unit, and that ${}_lR_r^X$ has the product topology. Then for every $p \in Z(R)^{(X)}$, $\zeta(p)$ is continuous, i.e., $\zeta(p) \in {}'({}_lR_r^X, \sqcap_X^\tau)'$. Whence, by Lemma 9, $\zeta \in \mathbf{Ab}(Z(R)^{(X)}, {}'({}_lR_r^X, \sqcap_X^\tau)')$ (where $\mathbf{Ab}$ denotes the category of abelian groups).*

Proof. This is clear from Lemma 10 since $\zeta(p) = \lambda(p)$ for every $p \in Z(R)^{(X)}$ or from Lemma 11 since $\zeta(p) = \rho(p)$ for every $p \in Z(R)^{(X)}$. □

Lemma 13. *Let us assume that (R,τ) is a Hausdorff topological ring with a unit, X is any set, and that ${}_lR_r^X$ has the product topology. For every $f \in R^X$, the family $(f(x)\delta_x)_{x\in X} = (\delta_x f(x))_{x\in X}$ is summable in $({}_lR_r^X, \sqcap_X^\tau)$ with sum f.*

Proof. By definition of the product topology, it is sufficient to prove that for every $x_0 \in X$ the family

$$({}_l\langle\delta_{x_0}, f(x)\delta_x\rangle)_{x\in X} = (f(x)\delta_x(x_0))_{x\in X}$$

is summable in (R,τ) with sum ${}_l\langle\delta_{x_0}, f\rangle = f(x_0)$, which is immediate. □

Remark 14. Lemma 13 legitimates the representation of a set-theoretic map $f\colon X \to R$ as the sum $\sum_{x\in X} f(x)\delta_x = \sum_{x\in X} \delta_x f(x)$ of a summable family. In particular when X is a free (commutative or not) monoid, then one recovers the usual formal power series (in commutative or not) variables.

Lemma 14. *Under the same assumptions as Lemma 13, if $\ell \in ({}_lR_r^X, \sqcap_X^\tau)'$, then*

$$D_\ell = \{\, x \in X \colon \ell(\delta_x) \text{ is right invertible in } R \,\}$$

is finite.

Proof. Since ℓ is a continuous (and a right R-linear) map, $(f(x)\delta_x)_{x\in X}$ is summable with sum f, and $f(x)\delta_x = \delta_x f(x)$ for each $x \in X$, then the family $(\ell(\delta_x)f(x))_{x\in X}$ is summable in (R,τ) with sum $\ell(f)$ for every $f \in R^X$. According to the properties of summability recalled in Section 2.4, for every neighborhood U of 0_R in (R,τ), $\ell(\delta_x)f(x)$ belongs to U for all but finitely many $x \in X$. Let us consider a particular choice for the map f, namely $f_0\colon X \to R$ such that $f_0(x) = 0$ for each $x \notin D_\ell$, and $f_0(x)$ is a chosen right inverse of $\ell(\delta_x)$ for each $x \in D_\ell$. Since R is assumed Hausdorff, there is some neighborhood U_0 of 0_R such that $1_R \notin U_0$. If D_ℓ is not finite, then $1_R = \ell(\delta_x)f_0(x) \notin U$ for every $x \in D_\ell$, which leads to a contradiction. □

Still by symmetry:

Lemma 15. *Under the same assumptions as Lemma 13, if* $\ell \in {}^{\backprime}({}_l R_r^X, \sqcap_X^\tau)$*, then*

$$G_\ell = \{\, x \in X : \ell(\delta_x) \textit{ is left invertible in } R \,\}$$

is finite.

Lemma 16. *Let* $(\mathbb{K}, \tau)$ *be a Hausdorff topological division ring, X be a set, and let us assume that* ${}_l\mathbb{K}_r^X$ *is equipped with the product topology* $\sqcap_X^\tau$*. Let* $\ell \in ({}_l\mathbb{K}_r^X)^*$ *(respectively,* $\ell \in {}^*({}_l\mathbb{K}_r^X)$*). If* $\ell \in ({}_l\mathbb{K}_r^X, \sqcap_X^\tau)'$ *(respectively,* $\ell \in {}^{\backprime}({}_l\mathbb{K}_r^X, \sqcap_X^\tau)$*), then* $\ell(\delta_x) = 0$ *for all but finitely many* $x \in X$*.*

Proof. By Lemma 14, the set $\{\, x \in X : \ell(\delta_x) \text{ is right invertible in } \mathbb{K} \,\} = \{\, x \in X : \ell(\delta_x) \neq 0 \,\}$ is finite. Using Lemma 15 one obtains the symmetric result. □

Lemma 17. *Under the same assumptions as Lemma 16,* λ *and* ρ *are onto.*

Proof. Let $\ell \in ({}_l\mathbb{K}_r^X, \sqcap_X^\tau)'$ be fixed. Let us define

$$\begin{array}{rccl} p_\ell \colon & X & \to & \mathbb{K} \\ & x & \mapsto & \ell(\delta_x). \end{array} \tag{11}$$

Of course $p_\ell \in \mathbb{K}^X$. But according to Lemma 16, p_ℓ actually belongs to $\mathbb{K}^{(X)}$. Let $f \in \mathbb{K}^X$. We have

$$\lambda(p_\ell)(f) = {}_l\langle p_\ell, f\rangle = \sum_{x\in X} p_\ell(x) f(x) = \sum_{x \in X} \ell(\delta_x) f(x) = \ell(f)$$

and then $\lambda(p_\ell) = \ell$. In a similar way one proves that ρ is onto. □

Finally,

Lemma 18. *Under the same assumptions as Lemma 16,* ζ *is onto.*

Proof. Let $\ell \in {}^{\backprime}({}_l\mathbb{K}_r^X, \sqcap_X^\tau)'$ be fixed, and let us define, as in the proof of Lemma 17, $p_\ell \colon x \in X \mapsto \ell(\delta_x)$. Of course, p_ℓ is finitely supported. For each $r \in \mathbb{K}$, one has $r\ell(\delta_x) = \ell(r\delta_x) = \ell(\delta_x r) = \ell(\delta_x)r$, hence $p_\ell \in Z(R)^{(X)}$. Finally $\ell(f) = \zeta(p_\ell)$. □

Now it is easy to conclude the proof of Theorem 5, since it follows directly from Lemmas 7, 10 and 17 for the right dual, from Lemmas 8, 11 and 17 for the left dual, and from Lemmas 9, 12 and 18 for the two-sided dual.

5. A few immediate remarks

5.1. Algebraic versus topological duals

In general, the algebraic duals ${}^*({}_l\mathbb{K}_r^X)$ and $({}_l\mathbb{K}_r^X)^*$ of ${}_l\mathbb{K}_r^X$ are not isomorphic to ${}_l\mathbb{K}_r^{(X)}$. Indeed, let for instance $(e_i)_{i\in I}$ be an algebraic right $\mathbb{K}$-basis of ${}_l\mathbb{K}^X$ (the existence of such a basis requires the axiom of choice for sets X of arbitrary large cardinal number). Therefore, every map $f \in \mathbb{K}^X$ may be (uniquely) written as a finite linear combination $\sum_{i\in I} e_i f_i$, with $f_i \in \mathbb{K}$ for each $i \in I$. Let us consider the unique right $\mathbb{K}$-linear map $\ell: {}_l\mathbb{K}^X \to \mathbb{K}$ such that $\ell(e_i) = 1$ for each $i \in I$. Thus, ℓ belongs to the right algebraic dual $({}_l\mathbb{K}_r^X)^*$ of ${}_l\mathbb{K}_r^X$. The family $(\delta_x)_{x\in X}$ is linearly independent in ${}_l\mathbb{K}^X$ (or in $\mathbb{K}_r^X$). Thus we may consider an algebraic right $\mathbb{K}$-basis of ${}_l\mathbb{K}^X$ that extends $(\delta_x)_{x\in X}$ (again by the axiom of choice, see for instance [9, Proposition 5.3, p. 335] or [2, Proposition 2.2.9, p. 57]). Now, the corresponding functional ℓ takes a non-zero value for each δ_x. Therefore, if X is infinite, then, according to Lemma 16, ℓ does not belong to the image of λ, or, in other terms, $\ell \notin ({}_l\mathbb{K}_r^X, \sqcap_X^\tau)'$, whatever is the Hausdorff division ring topology τ on $\mathbb{K}$. In particular, whenever $(\mathbb{K}, \tau)$ is a Hausdorff topological division ring, ${}_l\mathbb{K}_r^X$ has the product topology, and X is infinite, then ℓ is discontinuous at zero (and thus on the whole ${}_l\mathbb{K}_r^X$). In brief, when X is infinite, λ (respectively, ρ) is not an isomorphism from ${}_l\mathbb{K}_r^X$ to $({}_l\mathbb{K}_r^X)^*$ (respectively, ${}^*({}_l\mathbb{K}_r^X)$), and $({}_l\mathbb{K}_r^X, \sqcap_X^\tau)'$ (respectively, ${}'({}_l\mathbb{K}_r^X, \sqcap_X^\tau)$) is a proper sub-space of $({}_l\mathbb{K}_r^X)^*$ (respectively, ${}^*({}_l\mathbb{K}_r^X)$).

5.2. Number of field topologies

A field topology may be either Hausdorff or the indiscrete one. It remains many degrees of freedom since actually it is known (see [11,15]) that every infinite field $\mathbb{K}$ has $2^{2^{|\mathbb{K}|}}$ distinct field topologies.

5.3. Formal power series

Let (R, d) be a discrete commutative ring, and let X be a set. Let X^* be the free monoid on the alphabet X, ϵ be the empty word, and $|\omega|$ be the length of a word $\omega \in X^*$. Let us define $\mathfrak{M}_{\geq n} = \{f \in R^X : \nu(f) \geq n\}$, $n \in \mathbb{N}$, where $\nu(f) = \inf\{\, n \in \mathbb{N} : \exists \omega \in X^*,\ |\omega| \neq n, \text{ and } f(\omega) \neq 0 \,\}$ for every non-zero $f \in R^X$ (the infimum being taken in $\mathbb{N} \cup \{\infty\}$, with $\infty > n$ for every $n \in \mathbb{N}$, so that $\nu(0) = \infty$). The set R^{X^*}, seen as the R-algebra $R\langle\langle X\rangle\rangle$ of formal power series in non-commutative variables, may be topologized (as a topological R-algebra – see [21] – and, therefore, as a topological R-module) by the decreasing filtration of ideals $\mathfrak{M}_{\geq n}$: this is an example of the so-called *Krull topology* (given a ring R together with a decreasing filtration $R = \mathfrak{m}_0 \supseteq \mathfrak{m}_1 \supseteq \mathfrak{m}_2 \supseteq \cdots$ by two-sided ideals, the Krull topology on R is defined by declaring the subsets $\mathfrak{m}_j$ to be a base for the open neighborhoods of zero; see [8]), which is Hausdorff, and is the usual topology considered for formal power series in combinatorics and algebra; in case X is reduced to a single element x, we recover the usual $\mathfrak{M}$-adic topology of $\mathbb{K}[[x]]$, where $\mathfrak{M} = \langle x\rangle$ is the principal ideal generated by x. Whenever X is finite, this Krull topology coincides[4] with the product topology with a discrete R. According to Theorem 5, in case where X is finite and R is a topological field $\mathbb{K}$, the topological dual of $\mathbb{K}\langle\langle X\rangle\rangle$ (which is also a linearly compact space; see Section 3) is the space of polynomials $\mathbb{K}\langle X\rangle$ in non-commutative variables.

5.4. Dual space of the total contracted algebra

Take any monoid with a zero (see [5]) with the finite decomposition property (see [4,17]), and let R be a commutative ring with a unit. Let us consider the total contracted R-algebra $R_0[[M]]$ of the monoid with zero M (see [17]) that consists, as an R-module, to $\{\, f \in R^M : f(0_M) = 0_R \,\}$, where 0_M is the zero of M, while 0_R is the zero of the ring R. It is clear that $R_0[[M]] \simeq R^{M_0}$ (as R-modules), with $M_0 = M \setminus \{0_M\}$. Now, let us assume that $\mathbb{K}$ is a Hausdorff topological field. The product topology on $\mathbb{K}^M$ induces, as sub-space topology, the product topology on $\mathbb{K}^{M_0}$, that corresponds to that of $\mathbb{K}_0[[M]]$. The topological dual of $\mathbb{K}^{M_0}$ being $\mathbb{K}^{(M_0)}$ it is easy to check that $(\mathbb{K}_0[[M]])'$ is isomorphic to the underlying vector space of the (usual) contracted algebra (see [5]) $\mathbb{K}_0[M]$ of the monoid with zero M.

[4] Observe that if X is infinite both topologies are distinct: for instance, let $(x_n)_{n\geq 0}$ be a sequence of pairwise distinct elements of X, then this family is easily seen to be summable in the product topology with R discrete, while it does not converge in the Krull topology.

5.5. Functional analysis

Theorem 5 may be applied for the discrete topology d on $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, but also for the usual topologies $\tau_{|\cdot|}$ of $\mathbb{R}$ and $\mathbb{C}$, in such a way that the topological dual spaces $(\mathbb{R}^X)'$ or $(\mathbb{C}^X)'$ for both the discrete topology and the topology induced by the (usual) absolute values on $\mathbb{R}$, $\mathbb{C}$ are identical since isomorphic to $\mathbb{R}^{(X)}$ or $\mathbb{C}^{(X)}$. Notice that $\mathbb{K}^X$ is a Fréchet space[5], i.e., a denumerable projective limit of Banach spaces or equivalently a locally convex and complete space with a denumerable family of seminorms (see [20]), real or complex depending on whether $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, when is considered the product topology relative to the absolute value and when X is countable, and as such allows functional analysis like, for instance, Banach–Steinhaus, open map and closed graph theorems that do not hold in the case of the same space with the product topology relative to a discrete $\mathbb{K}$.

6. A category-theoretic recasting of Theorem 5

Our objective is to prove that Theorem 5 actually provides an equivalence of categories of vector spaces, that extends the equivalence between (the opposite category of) vector spaces and linearly compact vector spaces [7,1]. The first task, after some recalls about category theory, consists in defining the relevant categories, and then, in a second times, to exhibit a natural equivalence between them.

6.1. Some basic category-theoretic notions

Recall from Section 2.1 that the category $\mathbf{C}^{\mathsf{op}}$ is the *opposite category* of $\mathbf{C}$: both categories share the same objects, and each $f \in \mathbf{C}(a, b)$ is a member of $\mathbf{C}^{\mathsf{op}}(b, a)$. The composition of morphisms in $\mathbf{C}^{\mathsf{op}}$ is the opposite of that in $\mathbf{C}$. A sub-category $\mathbf{C}$ of $\mathbf{D}$ is said to be *full* when for all objects a, b of $\mathbf{C}$, $\mathbf{C}(a, b) = \mathbf{D}(a, b)$. A set O of objects in a category $\mathbf{D}$ determines a unique full sub-category, say $\mathbf{C}_O$, of $\mathbf{D}$, namely the *full sub-category of* $\mathbf{D}$ *spanned by* O, such that for all $a, b \in O$, $\mathbf{C}_O(a, b) = \mathbf{D}(a, b)$.

6.2. The category of double vector spaces

Let R be a (not necessarily commutative) ring with a unit. Let M be a R-R-bimodule such that ${}_l|M|$ is a free left R-module. Let X be a basis of this module in such a way ${}_l|M| \simeq {}_lR^{(X)}$ (as left R-modules). In general that does not imply that $M \simeq {}_lR_r^{(X)}$ as a bimodule (because the right action of R on M does not necessarily coincides with that on ${}_lR_r^{(X)}$ under the isomorphism).

Definition 2. Under the above assumptions one says that X is a *double R-basis* for the bimodule M whenever for each $r \in R$ and each $x \in X$, $rx = xr$.

Remark 15. Even if a R-R-bimodule M, which is free as a left R-module, admits a double R-basis X, this does not imply that for every $v \in M$, $r \in R$, $rv = vr$. (E.g., let $p \in R^{(X)}$, then in general $(rp)(x) = rp(x) \neq p(x)r$, $x \in X$, $r \in R$.)

Example 1. The set $\{\, \delta_x : x \in X \,\}$ is a double R-basis for ${}_lR_r^{(X)}$.

When X is a double R-basis for M, then it is quite obvious that $|M|_r \simeq R_r^{(X)}$ (as right R-modules), hence $|M|_r$ is also a free right R-module, and X is also a basis for it. Whence, $M \simeq {}_lR_r^{(X)}$ (as R-R-bimodules).

[5] It is not a Banach space when X is infinite.

There even exists a canonical isomorphism $\phi_X : M \to {}_l R_r^{(X)}$ which is uniquely determined by the relations $\phi_X(x) = \delta_x$, $x \in X$.

Following the terminology from [18] one introduces the following definition.

Definition 3. Let M be a R-R-bimodule. It is said to be a *R-double module* if it admits a double R-basis. When R is a division ring, then it is called a *R-double vector space*.

Remark 16. In general not all the bases of an R-double module are double bases. E.g., take any non-commutative division ring $\mathbb{K}$ (i.e., there are two elements r, s such that $rs \neq sr$ or in other terms the center $Z(\mathbb{K})$ of $\mathbb{K}$ is not $\mathbb{K}$ itself). Then, for each $X \neq \emptyset$, $\{ s\delta_x : x \in X \}$ is a basis of ${}_l R^{(X)}$ (or of $R_r^{(X)}$) which is not a double R-basis (since $rs\delta_x = \delta_x rs \neq \delta_x sr$).

One defines the full sub-category ${}_R\mathbf{DblMod}$ of ${}_R\mathbf{Bimod}_R$ spanned by those R-R-bimodules which are R-double modules. According to the above discussion, each object of ${}_R\mathbf{DblMod}$ is essentially of the form ${}_l R_r^{(X)}$. When R is a division ring, then ${}_R\mathbf{DblVect}$ is the full sub-category of all R-double vector spaces among the R-R-bi-vector spaces.

Remark 17. When $\mathbb{K}$ is a field, then any basis of a $\mathbb{K}$-vector space is automatically a double basis. Hence ${}_{\mathbb{K}}\mathbf{Vect} = {}_{\mathbb{K}}\mathbf{DblVect}$.

Remark 18. Let us assume that $\mathbb{K}$ is a division ring, and let V be a $\mathbb{K}$-$\mathbb{K}$-bi-vector space. Then, both ${}_l|V|$ and $|V|_r$ admit a $\mathbb{K}$-basis. But it may happen that V has no double $\mathbb{K}$-basis at all. In [10, p. 158], N. Jacobson gives an example of a bi-vector space over a field which, as a left vector space, has dimension 2, while, as a right vector space, has dimension 3. Hence there is no hope for it to admit a double basis.

One readily observes that any free left (or right) module over a commutative ring R may be seen as an R-double module when equipped with the same R-actions on the left and on the right.

6.3. Topologically-double vector spaces

Let (R, τ) be a topological ring. Let X be any set. One already knows that the bimodule ${}_l R_r^X$ admits a somewhat natural topology compatible with its bimodule structure, namely the *product topology* $\sqcap_X^\tau$, i.e., the coarsest topology β that makes continuous the projections $\pi_x : ({}_l R_r^X, \beta) \to (R, \tau)$, $f \mapsto f(x)$. (Because π_x is a bimodule map, for each x, the product topology is a topology of R-R-bimodule.) One observes that the set $\{ \delta_x : x \in X \}$, while not being in general an algebraic basis of the left, or the right, module structure on R^X, plays a similar role (see Lemma 13 and Remark 14), and, furthermore, commutes to the actions: $r\delta_x = \delta_x r$, $r \in R$, $x \in X$.

Let (M, σ) be a topological (R, τ)-(R, τ)-bimodule. It is said to be a *topologically-double (R, τ)-module* whenever there exists some set X such that $(M, \sigma) \simeq ({}_l R_r^X, \sqcap_X^\tau)$ as topological R-R-bimodules (hence they are both algebraically isomorphic and homeomorphic). Such a set X is referred to as a *topological double R-basis* (even it is not assumed to be a subset of M).

One defines the category ${}_{(R,\tau)}\mathbf{TopDblMod}$ of all topologically-double (R, τ)-modules, whose morphisms are those left and right R-linear maps which are also continuous. When R is a division ring, one may consider the category ${}_{(R,\tau)}\mathbf{TopDblVect}$ of all *topologically-double (R, τ)-vector spaces*.

Lemma 19. *The left and the right topological duals, ${}'(V, \sigma)$ and $(V, \sigma)'$, of a topologically-double $(\mathbb{K}, \tau)$-vector space (V, σ) are double $\mathbb{K}$-vector spaces.*

Proof. Let us prove the case of the left topological dual (the proof for the right topological dual is left to the reader). Let X be a topological double basis for (V,σ), and let $\Psi_X\colon (V,\sigma) \simeq ({}_l\mathbb{K}_r^X, \sqcap_X^\tau)$ be an isomorphism of topological $(\mathbb{K},\tau)$-$(\mathbb{K},\tau)$-bi-vector spaces (so both an algebraic isomorphism and a homeomorphism). Thus, because ${}^\backprime(-)$ is a functor, ${}^\backprime\Psi_X\colon {}^\backprime({}_l\mathbb{K}_r^X, \sqcap_X^\tau) \simeq {}^\backprime(V,\sigma)$ is an isomorphism of $\mathbb{K}$-$\mathbb{K}$-bi-vector spaces. According to Theorem 5, ${}^\backprime\Psi_X \circ \rho\colon {}_l\mathbb{K}_r^{(X)} \to {}^\backprime(V,\sigma)$ is also an isomorphism of $\mathbb{K}$-$\mathbb{K}$-bi-vector spaces. In particular, $\{ {}^\backprime\Psi_X(\rho(\delta_x))\colon x \in X \}$ is a left (or a right) basis of ${}^\backprime(V,\sigma)$. It remains to check that it is a double basis. Let $r \in \mathbb{K}$ and $v \in V$. Then, $(r \cdot {}^\backprime\Psi_X(\rho(\delta_x)))(v) = ({}^\backprime\Psi_X(\rho(\delta_x)))(vr) = \rho(\delta_x)(\Psi_X(vr)) = \langle \Psi_X(vr), \delta_x\rangle_r = \langle \Psi_X(v)r, \delta_x\rangle_r = \langle \Psi_X(v), r\delta_x\rangle_r = \langle \Psi_X(v), \delta_x r\rangle_r = \langle \Psi_X(v), \delta_x\rangle_r r = ({}^\backprime\Psi_X(\rho(\delta_x)) \cdot r)(v)$. (In the case of the right topological dual, one gets $\{ \Psi_X'(\lambda(\delta_x))\colon x \in X \}$ as a double basis for $(V,\sigma)'$.) □

The above Lemma 19 provides at once functors

$$ {}^\backprime(-)\colon {}_{(\mathbb{K},\tau)}\mathbf{TopDblVect}^{\mathsf{op}} \to {}_{\mathbb{K}}\mathbf{DblVect} $$

and

$$ (-)'\colon {}_{(\mathbb{K},\tau)}\mathbf{TopDblVect}^{\mathsf{op}} \to {}_{\mathbb{K}}\mathbf{DblVect}. $$

6.4. A topology on the algebraic duals

First of all one readily observes that one gets the following dual functors by restricting the domain category.

$$ {}^*(-)\colon {}_R\mathbf{DblMod}_R^{\mathsf{op}} \to {}_R\mathbf{Bimod}_R, $$
$$ (-)^*\colon {}_R\mathbf{DblMod}_R^{\mathsf{op}} \to {}_R\mathbf{Bimod}_R $$

and

$$ {}^*(-)^*\colon {}_R\mathbf{DblMod}_R^{\mathsf{op}} \to \mathbf{Ab}. $$

Similarly, assuming that R is a division ring, one gets

$$ {}^*(-)\colon {}_R\mathbf{DblVect}_R^{\mathsf{op}} \to {}_R\mathbf{Bivect}_R, $$
$$ (-)^*\colon {}_R\mathbf{DblVect}_R^{\mathsf{op}} \to {}_R\mathbf{Bivect}_R $$

and

$$ {}^*(-)^*\colon {}_R\mathbf{DblVect}_R^{\mathsf{op}} \to \mathbf{Ab}. $$

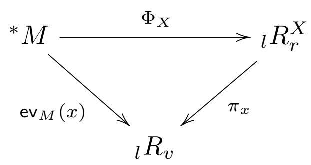
Now, let M be an R-double module. Hence there exists some set $X \subseteq M$ which is a double R-basis for M. Since X is, in particular, a basis of left module, hence $M \simeq {}_lR^{(X)}$, one has $|{}^*M| = {}_R\mathbf{Mod}({}_l|M|, {}_lR) \simeq R^X$, isomorphic as abelian groups, with a canonical isomorphism: $\Phi_X\colon \ell \mapsto (x \mapsto \ell(x))$.

Let $v \in M$, and let us introduce the *evaluation map at* v as $\mathsf{ev}_M(v)\colon \ell \mapsto \ell(v)$ from $|{}^*M|$ to R. It is automatically right R-linear even when M is just a R-R-bimodule, and not an R-double module. Indeed, $\mathsf{ev}_M(v)(\ell \cdot r)(x) = (\ell \cdot r)(x) = \ell(x)r = (\mathsf{ev}_M(v)(\ell))r$. Now, if v commutes to the left and right actions, i.e., $rv = vr$ for all $r \in R$, then $\mathsf{ev}_M(v)$ also is left R-linear since $\mathsf{ev}_M(v)(r \cdot \ell) = (r \cdot \ell)(v) = \ell(vr) = \ell(rv) = r\ell(v) = r(\mathsf{ev}_M(v)(\ell))$. Whence $\{ \mathsf{ev}_M(x)\colon x \in X \}$ is a set of R-R-bimodule maps from ${}^*M$ to ${}_lR_r$.

Let us assume that (R,τ) is a topological ring. One equips the R-R-bimodule ${}^*M$ with the initial topology Π_X^τ that makes continuous the evaluation maps $\mathsf{ev}_M(x)\colon \ell \mapsto \ell(x)$, $x \in X$, so that one obtains a topological

R-R-bimodule (see Section 2.3.2). This topology is Hausdorff as soon as (R, τ) is a Hausdorff topological ring (because the evaluation maps at members of X separate the points of ${}^*M$).

The isomorphism Φ_X lifts to an isomorphism of R-R-bimodules ${}^*M \simeq {}_lR_r^X$. (Indeed, $\Phi_X(r \cdot \ell)(x) = (r \cdot \ell)(x) = \ell(xr) = \ell(rx) = r\ell(x) = r((\Phi_X(\ell))(x))$, and $\Phi_X(\ell \cdot r)(x) = \ell(x)r = (\Phi_X(\ell)(x))r$.) Under this isomorphism $\mathsf{ev}_M(x)$ corresponds to the canonical projection $\pi_x \colon {}_lR_r^X \to {}_lR_r$ given by $\pi_x(f) = f(x)$. More precisely, this means that the following diagram commutes for each $x \in X$.



(12)

It then follows that the topology Π_X^τ on ${}^*M$ corresponds, under Φ_X, to the product topology on ${}_lR_r^X$, and Φ_X turns to be also a homeomorphism from $({}^*M, \Pi_X^\tau)$ onto $({}_lR_r^X, \sqcap_X^\tau)$ with the product topology. Therefore, $({}^*M, \Pi_X^\tau)$ is a topologically-double (R, τ)-module.

Lemma 20. *The topology Π_X^τ on ${}^*M$ does not depend on the choice of the double R-basis X, i.e., for any double R-basis Y of V, $\Pi_X^\tau = \Pi_Y^\tau$.*

Proof. It suffices to prove that for each $y \in Y$, $\mathsf{ev}_M(y)$ is continuous from $({}^*M, \Pi_X^\tau)$ to (R, τ), which amounts to mean that $\Pi_Y^\tau \subseteq \Pi_X^\tau$ by definition of an initial topology. Since X is a basis of, say, the left module M, one has $y = \sum_{x \in X_y} y_x x$, $y_x \in R$, and X_y is a finite subset of X. Hence $\mathsf{ev}_M(y) = \sum_{x \in X_y} y_x \mathsf{ev}_M(x)$. It then obviously follows that $\mathsf{ev}_M(y)$ is continuous (since any $\mathsf{ev}_M(x)$ is). □

Remark 19. The above lemma holds even if R does not satisfy the left (or right) invariant basis number property (see e.g. [2]), which means that every finite bases of a left (or right) free module share the same cardinality. (Infinite bases of a left, or a right, free module, over any non-zero ring, always have the same cardinality as it is shown in [2, Proposition 2.2.8].)

Definition 4. Let (R, τ) be a topological ring and let M be an R-double module. One denotes by Π_M^τ the topology of R-R-bimodule on ${}^*M$ which is equal, after Lemma 20, to any initial topology Π_X^τ for X a double R-basis of M.

Let (R, τ) be a topological ring, and let M, N be two R-double modules. Let $\phi \in {}_R\mathbf{DblMod}(M, N) = {}_R\mathbf{Bimod}_R(M, N)$. Then, one can prove that ${}^*\phi \colon {}^*N \to {}^*M$, defined by ${}^*\phi(\ell) = \ell \circ \phi$, is continuous with respect to the topologies Π_N^τ and Π_M^τ. By definition of the initial topology Π_M^τ (and according to Lemma 20), for any double R-basis X of M, one needs to show that $\mathsf{ev}_M(x) \circ \phi \colon {}^*N \to R$ is continuous for all $x \in X$. Let Y be any double R-basis of N. Then, $\phi(x) = \sum_{y \in Y_0} r_y y$ (where Y_0 is a finite subset of Y). Moreover, $\mathsf{ev}_M(x) \circ \phi = \mathsf{ev}_N(\phi(x)) = \sum_{y \in Y_0} r_y \mathsf{ev}_N(y)$. Hence, because $\mathsf{ev}_N(y)$ is continuous for all $y \in Y$, $\mathsf{ev}_M(x) \circ \phi$ is also continuous. Whence ${}^*(-)$ lifts to a functor from ${}_R\mathbf{DblMod}^{\mathsf{op}}$ to ${}_{(R,\tau)}\mathbf{TopDblMod}$. In a symmetrical way (the details are left to the reader) one also gets a functor $(-)^* \colon {}_R\mathbf{DblMod}^{\mathsf{op}} \to {}_{(R,\tau)}\mathbf{TopDblMod}$.

6.5. An equivalence of categories

Let $F, G \colon \mathbf{C} \to \mathbf{D}$ be two functors. A *natural transformation* $\alpha \colon F \Rightarrow G$ is a family $(\alpha_c)_c$ of arrows $\alpha_c \colon F(c) \to G(c)$ of $\mathbf{D}$ indexed by objects of $\mathbf{C}$ such that the following diagram commutes for each objects c_1, c_2 of $\mathbf{C}$ and each arrow $f \in \mathbf{C}(c_1, c_2)$.

$$\begin{array}{ccc} F(c_1) & \xrightarrow{\alpha_{c_1}} & G(c_1) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(c_2) & \xrightarrow[\alpha_{c_2}]{} & G(c_2) \end{array} \tag{13}$$

In case all components α_c of α are isomorphisms (i.e., morphisms f in $\mathbf{D}$ with a left and right inverse f^{-1}), then α is referred to as a *natural isomorphism*, and one denotes it by $\alpha\colon F \simeq G$. One observes that if $\alpha\colon F \simeq G$ is a natural isomorphism, then $\alpha^{-1}\colon G \simeq F$, where α^{-1} is the natural transformation with components α_c^{-1}.

One also recalls from [14] that a functor $F\colon \mathbf{C} \to \mathbf{D}$ is an *equivalence of categories*, and the categories $\mathbf{C}$ and $\mathbf{D}$ are *equivalent*, when there is a functor $G\colon \mathbf{D} \to \mathbf{C}$ and natural isomorphisms $F \circ G \simeq I_{\mathbf{D}}$ and $G \circ F \simeq I_{\mathbf{C}}$, where $I_{\mathbf{C}}$ is the *identity functor* of $\mathbf{C}$ (i.e., $I_{\mathbf{C}}(c) = c$ for each object c of $\mathbf{C}$ and $I_{\mathbf{C}}(f) = f$ for each arrow f of $\mathbf{C}$), and similarly for $\mathbf{D}$.

Finally one mentions the following obvious fact: a functor $F\colon \mathbf{C}^{\mathsf{op}} \to \mathbf{D}$ may be equally seen as a functor $F\colon \mathbf{C} \to \mathbf{D}^{\mathsf{op}}$. Having that in mind, for $(\mathbb{K}, \tau)$ a Hausdorff topological division ring, from Sections 6.3 and 6.4, one has two pairs of functors, namely,

$$\begin{aligned} (-)'\colon {}_{(\mathbb{K},\tau)}\mathbf{TopDblVect} &\to {}_{\mathbb{K}}\mathbf{DblVect}^{\mathsf{op}}, \\ {}^{*}(-)\colon {}_{\mathbb{K}}\mathbf{DblVect}^{\mathsf{op}} &\to {}_{(\mathbb{K},\tau)}\mathbf{TopDblVect} \end{aligned}$$

and

$$\begin{aligned} {}^{\backprime}(-)\colon {}_{(\mathbb{K},\tau)}\mathbf{TopDblVect} &\to {}_{\mathbb{K}}\mathbf{DblVect}^{\mathsf{op}}, \\ (-)^{*}\colon {}_{\mathbb{K}}\mathbf{DblVect}^{\mathsf{op}} &\to {}_{(\mathbb{K},\tau)}\mathbf{TopDblVect}. \end{aligned}$$

Our next goal is to prove that each pair is part of an equivalence of categories. Let us only treat in details the case of the first pair (the second one may be carried on by symmetry). Let V be a double $\mathbb{K}$-vector space, and let us introduce $\Gamma_V\colon V \to ({}^*V, \Pi_V^\tau)'$ by

$$\Gamma_V(v)(\ell) = \ell(v)$$

for each $v \in V$ and each $\ell \in {}^*V$. (Actually one should first define Γ_V as a map from V to $V^{{}^*V}$, and proves that its range is a part of $({}^*V, \Pi_V^\tau)'$. For each $v \in V$, $\ell \in {}^*V$, and $r \in \mathbb{K}$, one has $\Gamma_V(v)(\ell \cdot r) = (\ell \cdot r)(v) = \ell(v)r = (\Gamma_V(v)(\ell))r$ which shows that $\Gamma_V(v) \in ({}^*V)^*$. Moreover $\Gamma_V(v)$ is continuous from $({}^*V, \Pi_V^\tau)$ to $(\mathbb{K}, \tau)$ because $\Gamma_V(v) = \mathsf{ev}_V(v)$, v may be written as a finite linear combination of members of a double basis of V, and the evaluation maps at these elements are continuous by definition of Π_V^τ. Hence $\Gamma_V(v) \in ({}^*V)'$.) Furthermore, Γ_V is left and right $\mathbb{K}$-linear. Indeed,

$$\begin{aligned} \Gamma_V(rv)(\ell) &= \ell(rv) \\ &= r\ell(v) \\ &= r(\Gamma_V(v)(\ell)) \\ &= (r \cdot \Gamma_V(v))(\ell) \end{aligned} \tag{14}$$

and

$$\begin{aligned}\Gamma_V(vr)(\ell) &= \ell(vr)\\ &= (r\cdot\ell)(v)\\ &= \Gamma_V(v)(r\cdot\ell)\\ &= (\Gamma_V(v)\cdot r)(\ell).\end{aligned}\tag{15}$$

The kernel of Γ_V is reduced to 0: let $v \in \ker\Gamma_V$, and let X be a double basis of V, then the canonical projection $\pi_x : V \to \mathbb{K}$, $x \in X$, belongs to ${}^*V$, so that $\Gamma_V(v)(\pi_x) = 0$ for all $x \in X$. This implies that $v = 0$. One now proves that Γ_V is onto. To see this it suffices to observe that, for each double basis X of V, the following diagram commutes in the category of $\mathbb{K}$-$\mathbb{K}$-vector spaces (and even in the category of double $\mathbb{K}$-vector spaces), and that the arrows ϕ_X (Section 6.2), λ (Section 4) and Φ'_X (Section 6.4) are isomorphisms (for the later because Φ_X is an isomorphism, and any functor preserves isomorphisms). Actually this directly proves that Γ_V is an isomorphism.

$$\begin{array}{ccc} V & \xrightarrow{\;\Gamma_V\;} & ({}^*V, \Pi^\tau_X)' \\ {\scriptstyle\phi_X}\big\downarrow & & \big\uparrow{\scriptstyle\Phi'_X} \\ {}_l\mathbb{K}_r^{(X)} & \xrightarrow[\;\lambda\;]{} & ({}_l\mathbb{K}_r^X, \Pi^\tau_X)' \end{array}\tag{16}$$

The above diagram commutes, because one has, for each $v \in V$, $\ell \in {}^*V$,

$$\begin{aligned}((\Phi'_X \circ \lambda \circ \phi)(v))(\ell) &= (\lambda(\phi(v)) \circ \Phi_X)(\ell)\\ &= {}_l\langle \phi_X(v), \Phi_X(\ell)\rangle.\end{aligned}\tag{17}$$

Now, $v = \sum_{x\in X} v_x x$ as a linear combination (since X is a left basis of V), hence one has

$$\begin{aligned}{}_l\langle \phi_X(v), \Phi_X(\ell)\rangle &= {}_l\langle \sum_{x\in X} v_x\delta_x, \sum_{x\in X} \ell(x)\delta_x\rangle \quad \text{(by definitions of } \phi_X \text{ and } \Phi_X)\\ &= \sum_{x\in X} v_x \ell(x).\end{aligned}\tag{18}$$

Moreover one also has $\Gamma_V(v)(\ell) = \Gamma_V(\sum_{x\in X} v_x x)(\ell) = \sum_{x\in X} v_x\Gamma_V(x)(\ell) = \sum_{x\in X} v_x\ell(x)$.

The fact that $(\Gamma_V)_V$ is a natural transformation follows from the equalities below. Let V, W be two double $\mathbb{K}$-vector spaces, and let $\phi : W \to V$ be a left and right linear map. For each $w \in W$ and $\ell \in {}^*V$,

$$((\Gamma_V \circ \phi)(w))(\ell) = \ell(\phi(w)),$$

while

$$\begin{aligned}((({}^*\phi)' \circ \Gamma_W)(w))(\ell) &= (\Gamma_W(w) \circ {}^*\phi)(\ell)\\ &= \Gamma_W(w)(\ell\circ\phi)\\ &= \ell(\phi(w)).\end{aligned}\tag{19}$$

Hence, $(\Gamma_V)_V : I_{{}_{\mathbb{K}}\mathbf{DblVect}^{\mathrm{op}}} \simeq ({}^*(-))'$ is a natural isomorphism.

Now, let (V, σ) be a topologically-double $(\mathbb{K}, \tau)$-vector space. Let us define the left and right $\mathbb{K}$-linear and continuous map

$$\Theta_{(V,\sigma)} : (V, \sigma) \to ({}^*((V, \sigma)'), \Pi^\tau_{(V,\sigma)'})$$

by

$$\Theta_{(V,\sigma)}(v)(\ell) = \ell(v)$$

for each $v \in V$, $\ell \in (V,\sigma)'$. (For each $v \in V$, $\Theta_{(V,\sigma)}$ is indeed left $\mathbb{K}$-linear because $\Theta_{(V,\sigma)}(v)(r \cdot \ell) = (r \cdot \ell)(v) = r(\ell(v)) = r(\Theta_{(V,\sigma)}(v)(\ell))$ for each $\ell \in (V,\sigma)'$ and each $r \in \mathbb{K}$, and, furthermore, Θ is indeed left and right linear since $\Theta_{(V,\sigma)}(rv)(\ell) = \ell(rv) = (\ell \cdot r)(v) = \Theta_{(V,\sigma)}(v)(\ell \cdot r) = (r \cdot \Theta_{(V,\sigma)}(v))(\ell)$, and $\Theta_{(V,\sigma)}(vr)(\ell) = \ell(vr) = \ell(v)r = (\Theta_{(V,\sigma)}(v)(\ell))r = (\Theta_{(V,\sigma)}(v) \cdot r)(\ell)$.) Finally $\Theta_{(V,\sigma)}$ is indeed continuous: let X be a topological double basis of $(V\sigma)$, then according to Lemma 19, $\{\, \Psi'_X(\lambda(\delta_x)) : x \in X \,\}$ is a double basis for $(V,\sigma)'$, where $\Psi_X : (V,\sigma) \simeq ({}_l\mathbb{K}_r^X, \sqcap_X^\tau)$. The topology $\Pi^\tau_{(V,\sigma)'}$ on ${}^*((V,\sigma)')$ is the initial topology that makes continuous the maps $\mathsf{ev}_{(V,\sigma)'}(\Psi'_X(\lambda(\delta_x))) = \mathsf{ev}_{(V,\sigma')}(\lambda(\delta_x) \circ \Psi_X) : (V,\sigma)' \to \mathbb{K}$, $x \in X$. One has $(\mathsf{ev}_{(V,\sigma')}(\lambda(\delta_x) \circ \Psi_X) \circ \Theta_{(V,\sigma)})(v) = \Theta_{(V,\sigma)}(v)(\lambda(\delta_x) \circ \Psi_X) = (\lambda(\delta_x))(\Psi_X(v)) = {}_l\langle \delta_x, \Psi_X(v) \rangle = (\pi_x \circ \Psi_X)(v)$. Since both Ψ_X and π_x, $x \in X$, are continuous, it follows that $\mathsf{ev}_{(V,\sigma')}(\lambda(\delta_x) \circ \Psi_X) \circ \Theta_{(V,\sigma)}$ also is, and by definition of the topology $\Pi^\tau_{(V,\sigma)'}$, $\Theta_{(V,\sigma)}$ itself is continuous.

$\Theta_{(V,\sigma)}$ is one-to-one: Let X be a topological double basis of (V,σ), and let $\Psi_X : (V,\sigma) \simeq ({}_l\mathbb{K}_r^X, \sqcap_X^\tau)$. Then, $\Psi'_X : ({}_l\mathbb{K}_r^X, \sqcap_X^\tau)' \simeq (V,\sigma)'$. Hence, according to Theorem 5, $\Psi'_X \circ \lambda : {}_l\mathbb{K}_r^{(X)} \simeq (V,\sigma)'$. Let $v \in \ker \Theta_{(V,\sigma)}$. Then, in particular, for all $x \in X$, $\Theta_{(V,\sigma)}(v)(\Psi'_X(\lambda(\delta_x))) = (\Psi'_X(\lambda(\delta_x)))(v) = \lambda(\delta_x)(\Psi_X(v)) = {}_l\langle \delta_x, \Psi_X(v) \rangle = 0$. Thus, $\Psi_X(v) = 0$ (by non-degeneracy of ${}_l\langle \cdot, \cdot \rangle$) so that $v = 0$.

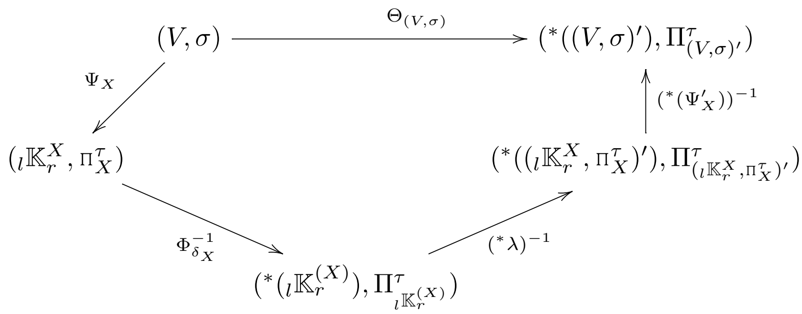
$\Theta_{(V,\sigma)}$ is onto: Let X be a topological double basis of (V,σ), and $\Psi_X : (V,\sigma) \simeq ({}_l\mathbb{K}_r^X, \sqcap_X^\tau)$. Following Section 6.4, because $\delta_X = \{\, \delta_x : x \in X \,\}$ is a double basis of ${}_l\mathbb{K}_r^{(X)}$, one has $\Phi_{\delta_X} : ({}^*({}_l\mathbb{K}_r^{(X)}), \Pi^\tau_{{}_l\mathbb{K}_r^{(X)}}) \simeq ({}_l\mathbb{K}_r^X, \sqcap_X^\tau)$. Finally, one has $\lambda : {}_l\mathbb{K}_r^{(X)} \simeq ({}_l\mathbb{K}_r^X, \sqcap_X^\tau)'$, and $\Psi'_X : ({}_l\mathbb{K}_r^X, \sqcap_X^\tau)' \simeq (V,\sigma)'$, so that

$${}^*\lambda : ({}^*(({}_l\mathbb{K}_r^X, \sqcap_X^\tau)'), \Pi^\tau_{({}_l\mathbb{K}_r^X, \sqcap_X^\tau)'}) \simeq ({}^*({}_l\mathbb{K}_r^{(X)}), \Pi^\tau_{{}_l\mathbb{K}_r^{(X)}})$$

and also

$${}^*(\Psi'_X) : ({}^*((V,\sigma)'), \Pi^\tau_{(V,\sigma)'}) \simeq ({}^*({}_l\mathbb{K}_r^{(X)}), \Pi^\tau_{{}_l\mathbb{K}_r^{(X)}}).$$

We now prove that the following diagram commutes implying that $\Theta_{(V,\sigma)}$ is actually an isomorphism (so it is onto) since all the other arrows are isomorphisms.



(20)

To prove the commutativity of the above diagram, let $v \in V$ and $\ell \in (V,\sigma)'$, and one computes the following:

$$\begin{aligned}({}^*(\lambda^{-1} \circ (\Psi_X^{-1})') \circ \Phi_{\delta_X}^{-1} \circ \Psi_X)(v)(\ell) &= (\Phi_{\delta_X}^{-1}(\Psi_X(v)))(\lambda^{-1}((\Psi_X^{-1})'(\ell))) \\ &= (\Phi_{\delta_X}^{-1}(\Psi_X(v)))(\lambda^{-1}(\ell \circ \Psi_X^{-1})) \\ &= (\Phi_{\delta_X}^{-1}(\Psi_X(v))) \left(\sum_{x \in X} \ell(\Psi_X^{-1}(\delta_x))\delta_x \right) \qquad (21)\end{aligned}$$

(sum with only finitely many non-zero terms since $\ell \circ \Psi_X^{-1} \in ({}_l\mathbb{K}_r^X, \sqcap_X^\tau)'$)

$$\begin{aligned}
&= \sum_{x \in X} \ell(\Psi_X^{-1}(\delta_x))\Phi_{\delta_X}^{-1}(\Psi_X(v))(\delta_x) \\
&= \ell\left(\sum_{x \in X} \Psi_X^{-1}(\delta_x)\Phi_{\delta_X}^{-1}(\Psi_X(v))(\delta_x)\right) \\
&= \ell(v) \\
&= \Theta_{(V,\sigma)}(v)(\ell).
\end{aligned} \tag{22}$$

Let us make explicit the penultimate equality. Since $\Psi_X(v) \in \mathbb{K}^X$, one has $\Psi_X(v) = \sum_{x \in X} \delta_x \Psi_X(v)(x)$ (by Lemma 13). Then, $v = \Psi_X^{-1}(\Psi_X(v)) = \sum_{x \in X} \Psi_X^{-1}(\delta_x)\Psi_X(v)(x)$ (by continuity), but $\Psi_X(v)(x) = \Phi_{\delta_X}^{-1}(\Psi_X(v))(\delta_x)$.

Finally, $(\Theta_{(V,\sigma)})_{(V,\sigma)}$ is a natural transformation from $I_{{}_{(\mathbb{K},\tau)}\mathbf{TopDblVect}}$ to ${}^*((-)')$, as it can be shown in the same way as $(\Gamma_V)_V$.

Hence, $(\Theta_{(V,\sigma)})_{(V,\sigma)}: I_{{}_{(\mathbb{K},\tau)}\mathbf{TopDblVect}} \simeq {}^*((-)')$ is a natural isomorphism.

The results from this section may be summarized in the following theorem which, thereby, is a category-theoretic reformulation of (a part of) Theorem 5.

Theorem 21. *For every Hausdorff topological division ring $(\mathbb{K}, \tau)$, the categories ${}_{\mathbb{K}}\mathbf{DblVect}^{\mathsf{op}}$ and ${}_{(\mathbb{K},\tau)}\mathbf{TopDblVect}$ are equivalent. In particular, for every Hausdorff division ring topologies τ, σ on a division ring $\mathbb{K}$, the categories ${}_{(\mathbb{K},\tau)}\mathbf{TopDblVect}$ and ${}_{(\mathbb{K},\sigma)}\mathbf{TopDblVect}$ are equivalent.*

In the case where $\mathbb{K}$ is a field, the category ${}_{\mathbb{K}}\mathbf{DblVect}$ is the same as the category ${}_{\mathbb{K}}\mathbf{Vect}$ of $\mathbb{K}$-vector spaces. A very particular instance of Theorem 21 is then the equivalence between ${}_{\mathbb{K}}\mathbf{Vect}$ and ${}_{(\mathbb{K},\mathsf{d})}\mathbf{TopDblVect}$ (where d is the discrete topology) which is precisely the equivalence between $\mathbb{K}$-vector spaces and linearly compact $\mathbb{K}$-vector spaces [1, Proposition 24.8, p. 107]. Furthermore this holds for all Hausdorff field topologies, and not only for the discrete one.

7. Some consequences of Theorem 5

7.1. A partial reciprocal to Theorem 5

We can deduce an immediate corollary of Theorem 5 which is a partial reciprocal to our main result.

Corollary 1. *Let $(\mathbb{K}, \tau)$ be a topological division ring, and let X be a set. Let us assume that ${}_l\mathbb{K}_r^X$ has the product topology $\sqcap_X^\tau$. Let us also assume that X is infinite. Then, ${}_l\mathbb{K}_r^{(X)}$ is isomorphic to $({}_l\mathbb{K}_r^X, \sqcap_X^\tau)'$ (respectively, ${}'({}_l\mathbb{K}_r^X, \sqcap_X^\tau)$) by λ (respectively, ρ), i.e., λ (respectively, ρ) is onto, if, and only if, $(\mathbb{K}, \tau)$ is Hausdorff.*

Proof. If $(\mathbb{K}, \tau)$ is Hausdorff, then according to Theorem 5 $({}_l\mathbb{K}_r^X, \sqcap_X^\tau)'$ (respectively, ${}'({}_l\mathbb{K}_r^X, \sqcap_X^\tau)$) is isomorphic to ${}_l\mathbb{K}_r^{(X)}$. Let us prove the converse assertion. Let us assume that $({}_l\mathbb{K}_r^X, \sqcap_X^\tau)'$ (respectively, ${}'({}_l\mathbb{K}_r^X, \sqcap_X^\tau)$) is isomorphic to ${}_l\mathbb{K}_r^{(X)}$. Now, let (R, t) be an indiscrete topological ring. Then, with respect to the trivial topology t on R, $({}_lR_r^X, \sqcap_{\mathsf{t}}^X)' = ({}_lR_r^X)^*$ (respectively, ${}'({}_lR_r^X, \sqcap_{\mathsf{t}}^X) = {}^*({}_lR_r^X)$). Because X is infinite, the division ring $\mathbb{K}$ cannot be indiscrete (see the above Section 5.1), i.e., $\tau \neq \mathsf{t}$. But ring topologies on a division ring (and in particular division ring topologies) may be either Hausdorff or the indiscrete one (see [21, Corollary 4.7, p. 25]). $\square$

7.2. On the consequences on continuous linear maps

As explained in the Introduction, the rigidity of the dual spaces with respect to the change of product topologies forces continuous linear maps (with respect to any of those topologies) to be represented by "row-finite" matrices as we now show.

Let $(\mathbb{K}, \tau)$ be a Hausdorff topological division ring, and let X, Y be two sets. As usually, we assume that $\mathbb{K}^Z$ has the product topology $\sqcap_Z^\tau$ for $Z \in \{X, Y\}$. The category of topological right (respectively, left, bi-) vector spaces over $(\mathbb{K}, \tau)$, with continuous and right (respectively, left, and left and right) linear maps, is denoted as usually by $\mathbf{TopVect}_{(\mathbb{K},\tau)}$ (respectively, ${}_{(\mathbb{K},\tau)}\mathbf{TopVect}$, ${}_{(\mathbb{K},\tau)}\mathbf{TopBivect}_{(\mathbb{K},\tau)}$).

Following Section 2.5.1, it is clear how to equip $\mathbf{TopVect}_{(\mathbb{K},\tau)}((V, \sigma), (W, \mu))$ and ${}_{(\mathbb{K},\tau)}\mathbf{TopVect}((V, \sigma), (W, \mu))$ with structures of $\mathbb{K}$-$\mathbb{K}$-bi-vector spaces, when both (V, σ) and (W, μ) are topological $(\mathbb{K}, \tau)$-$(\mathbb{K}, \tau)$-bi-vector spaces. Of course, ${}_{(\mathbb{K},\tau)}\mathbf{TopBivect}_{(\mathbb{K},\tau)}((V, \sigma), (W, \mu))$ is just an abelian group.

We denote by ${}_l\mathbb{K}_r^{Y\times(X)}$ the $\mathbb{K}$-$\mathbb{K}$-bi-vector space (under component-wise operations) of all maps $M : Y \times X \to \mathbb{K}$ such that for each $y \in Y$, the set $\{ x \in X : M(y, x) \neq 0 \}$ is finite (in particular $\mathbb{K}^{Y\times(X)}$ is its underlying abelian group). If $X = Y = \mathbb{N}$, then is recovered the usual notion of row-finite matrices (see [6]). One may also define ${}_l\mathbb{K}_r^{(X)\times Y}$ as the bi-vector space of all maps $M : X \times Y \to \mathbb{K}$ such that for each $y \in Y$, the set $\{ x \in X : M(x, y) \neq 0 \}$ is finite. Of course, ${}_l\mathbb{K}_r^{Y\times(X)}$ and ${}_l\mathbb{K}_r^{(X)\times Y}$ are isomorphic under the *transpose map* $M^t(x, y) = M(y, x)$, $x \in X$, $y \in Y$, for $M \in \mathbb{K}^{Y\times(X)}$. Recall also that if $p \in \mathbb{K}^{(X)}$, then its support is given by

$$\mathsf{supp}(p) = \{ x \in X : p(x) \neq 0 \}. \tag{23}$$

Let $\phi \in \mathbf{TopVect}_{(\mathbb{K},\tau)}(({}_l\mathbb{K}_r^X, \sqcap_X^\tau), ({}_l\mathbb{K}_r^Y, \sqcap_Y^\tau))$. We define the following map:

$$\begin{array}{rccl} M_\phi : & Y \times X & \to & \mathbb{K} \\ & (y, x) & \mapsto & {}_l\langle \delta_y, \phi(\delta_x) \rangle. \end{array} \tag{24}$$

Lemma 22. *For each $\phi \in \mathbf{TopVect}_{(\mathbb{K},\tau)}(({}_l\mathbb{K}_r^X, \sqcap_X^\tau), ({}_l\mathbb{K}_r^Y, \sqcap_Y^\tau))$, $M_\phi \in \mathbb{K}^{Y\times(X)}$, and the map $\phi \mapsto M_\phi$ is one-to-one, and is left and right $\mathbb{K}$-linear.*

Proof. For every $x \in X$, the map

$$\begin{array}{rcl} \mathbb{K}^X & \to & \mathbb{K} \\ f & \mapsto & {}_l\langle \delta_y, \phi(f) \rangle = \phi(f)(\delta_y) \end{array} \tag{25}$$

is an element of $({}_l\mathbb{K}_r^X, \sqcap_X^\tau)'$ (because it is the composition of ϕ with the projection onto $\mathbb{K}\delta_y$, and both of them are right linear maps). According to Theorem 5, there is one, and only one, $p_{\phi,y} \in \mathbb{K}^{(X)}$ such that for every $f \in \mathbb{K}^X$, ${}_l\langle p_{\phi,y}, f \rangle = \lambda(p_{\phi,y})(f) = {}_l\langle \delta_y, \phi(f) \rangle$. In particular, for every $x \in X$, $p_{\phi,y}(x) = {}_l\langle p_{\phi,y}, \delta_x \rangle = {}_l\langle \delta_y, \phi(\delta_x) \rangle$, hence $\{ x \in X : \langle \delta_y, \phi(\delta_x) \rangle \neq 0 \} = \mathsf{supp}(p_{\phi,y})$, and then $M_\phi \in \mathbb{K}^{Y\times(X)}$. Now let us assume that $M_\phi = M_{\phi'}$, then for every $(y, x) \in Y \times X$, ${}_l\langle \delta_y, \phi(\delta_x) \rangle = {}_l\langle \delta_y, \phi'(\delta_x) \rangle$. Then, by bilinearity, $\phi(\delta_x)(y) - \phi'(\delta_x)(y) = {}_l\langle \delta_y, \phi(\delta_x) - \phi'(\delta_x) \rangle = 0$. Since this last equality holds for every $y \in Y$, $\phi(\delta_x) = \phi'(\delta_x)$ for every $x \in X$. Now, let $f \in \mathbb{K}^X$, since $f = \sum_{x\in X} \delta_x f(x)$ (sum of a summable family), by continuity, $\phi(f) = \sum_{x\in X} \phi(\delta_x) f(x) = \sum_{x\in X} \phi'(\delta_x) f(x) = \phi'(f)$. It remains to check that $\phi \mapsto M_\phi$ is left and right linear (additivity is obvious). Let $r \in \mathbb{K}$, $(y, x) \in Y \times X$. One has

$$\begin{aligned} M_{r\cdot\phi}(y,x) &= {}_l\langle \delta_y, (r\cdot\phi)(\delta_x)\rangle \\ &= {}_l\langle \delta_y, r\phi(\delta_x)\rangle \\ &= {}_l\langle \delta_y r, \phi(\delta_x)\rangle \\ &= {}_l\langle r\delta_y, \phi(\delta_x)\rangle \\ &= r{}_l\langle \delta_y, \phi(\delta_x)\rangle \\ &= r(M_\phi(y,x)) \end{aligned} \tag{26}$$

and

$$\begin{aligned} M_{\phi\cdot r}(y,x) &= {}_l\langle \delta_y, (\phi\cdot r)(\delta_x)\rangle \\ &= {}_l\langle \delta_y, \phi(r\delta_x)\rangle \\ &= {}_l\langle \delta_y, \phi(\delta_x r)\rangle \\ &= {}_l\langle \delta_y, \phi(\delta_x) r\rangle \\ &= {}_l\langle \delta_y, \phi(\delta_x)\rangle r \\ &= (M_\phi(y,x))r. \qquad \square \end{aligned} \tag{27}$$

Remark 20. Given $\phi \in {}_{(\mathbb{K},\tau)}\mathbf{TopVect}(({}_l\mathbb{K}_r^X, \sqcap_X^\tau), ({}_l\mathbb{K}_r^Y, \sqcap_Y^\tau))$, one may define ${}_\phi M\colon X\times Y \to \mathbb{K}$ by ${}_\phi M(x,y) = \langle \phi(\delta_x), \delta_y\rangle_r$. The result corresponding to Lemma 22 in this case states that ${}_\phi M \in \mathbb{K}^{(X)\times Y}$ and the map $\phi \mapsto {}_\phi M$ is one-to-one, and linear on both sides. It is also possible to establish a "two-sided" version. Let $\phi \in {}_{(\mathbb{K},\tau)}\mathbf{TopBivect}_{(\mathbb{K},\tau)}(({}_l\mathbb{K}_r^X, \sqcap_X^\tau), ({}_l\mathbb{K}_r^Y, \sqcap_Y^\tau))$. Then, ${}_\phi M(x,y) = M_\phi^t(x,y) \in Z(\mathbb{K})$ for every $(x,y) \in X\times Y$. Indeed, the map $f \in \mathbb{K}^X \mapsto {}_l\langle \delta_y, \phi(f)\rangle = \phi(f)(\delta_y) = (\pi_y \circ \phi)(f) \in \mathbb{K}$ is a member of ${}^\backprime({}_l\mathbb{K}_r^X, \sqcap_X^\tau)'$. Thus, by Theorem 5, there exists a unique $p_{\phi,y} \in Z(\mathbb{K})^{(X)}$ such that ${}_l\langle p_{y,\phi}, f\rangle = {}_l\langle \delta_y, \phi(f)\rangle$. In particular, ${}_l\langle p_{y,\phi}, \delta_x\rangle = {}_l\langle \delta_y, \phi(\delta_x)\rangle = M_\phi(y,x) \in Z(\mathbb{K})$. Moreover, $M_\phi(y,x) = {}_l\langle p_{y,\phi}, \delta_x\rangle = \langle \delta_x, p_{y,\phi}\rangle_r = {}_\phi M(x,y)$. It follows that $M_\phi^t = {}_\phi M \in Z(\mathbb{K})^{(X)\times Y}$. Of course, $\phi \mapsto {}_\phi M$ and $\phi \mapsto M_\phi$ are one-to-one homomorphisms of abelian groups.

Theorem 23. *The $\mathbb{K}$-$\mathbb{K}$-bi-vector spaces $\mathbf{TopVect}_{(\mathbb{K},\tau)}(({}_l\mathbb{K}_r^X, \sqcap_X^\tau), ({}_l\mathbb{K}_r^Y, \sqcap_Y^\tau))$ and ${}_l\mathbb{K}_r^{Y\times(X)}$ are isomorphic. More precisely the map $\phi \mapsto M_\phi$ of Lemma 22 is also onto.*

Proof. Let $M \in \mathbb{K}^{Y\times(X)}$. Let us define $\psi_M\colon \mathbb{K}^X \to \mathbb{K}^Y$ by

$$\psi_M(f) = \psi_M(\sum_{x\in X} f(x)\delta_x) = \sum_{y\in Y}\left(\sum_{x\in X} M(y,x)f(x)\right)\delta_y,$$

i.e., $\psi_M(f)(y) = \sum_{x\in X} M(y,x)f(x)$. (Clearly the second sum on $x \in X$ has only finitely many non-zero terms since $M \in \mathbb{K}^{Y\times(X)}$, and therefore is defined in $\mathbb{K}$.) The map ψ_M is clearly right $\mathbb{K}$-linear. Let us prove that it is continuous. By definition of the product topology, it is sufficient to prove that for every $y \in Y$, $\ell_{M,y}\colon \mathbb{K}^X \to \mathbb{K}$, defined by $\ell_{M,y}(f) = {}_l\langle \delta_y, \psi_M(f)\rangle = \sum_{x\in X} M(y,x)f(x)$, is continuous and this is true since $\ell_{M,y}$ is a finite sum of scalar multiples of projections, so $\psi_M \in \mathbf{TopVect}_{(\mathbb{K},\tau)}(({}_l\mathbb{K}_r^X, \sqcap_X^\tau), ({}_l\mathbb{K}_r^Y, \sqcap_Y^\tau))$. Finally we prove that $M_{\psi_M} = M$. Let $(y,x) \in Y\times X$, we have

$$\begin{aligned}M_{\psi_M}(y,x) &= {}_l\langle \delta_y, \psi_M(\delta_x)\rangle \\ &= {}_l\langle \delta_y, \sum_{y'\in Y}\left(\sum_{z\in X} M(y',z)\delta_x(z)\right)\delta_{y'}\rangle \\ &= \sum_{z\in X} M(y,z)\delta_x(z) \\ &= M(y,x). \end{aligned} \tag{28}$$

The map $\phi \mapsto M_\phi$ thus is onto, and, by Lemma 22, it is a bijection. □

Remark 21. Once again, Theorem 23 admits a left version: The $\mathbb{K}$-$\mathbb{K}$-bi-vector spaces ${}_{(\mathbb{K},\tau)}\mathbf{TopVect}(({}_l\mathbb{K}_r^X, \sqcap_X^\tau), ({}_l\mathbb{K}_r^Y, \sqcap_Y^\tau))$ and ${}_l\mathbb{K}_r^{(X)\times Y}$ are isomorphic under the map $\phi \mapsto {}_\phi M$. (To prove this statement it suffices, as in the proof of Theorem 23, to show that the above map is onto by defining ${}_M\psi\colon \mathbb{K}^X \to \mathbb{K}^Y$ as ${}_M\psi(f) = \sum_{y\in Y}\left(\sum_{x\in X} f(x)M(x,y)\right)\delta_y$.) In case where $M \in Z(\mathbb{K})^{(X)\times Y}$, ${}_M\psi = \psi_{M^t}$, so that ${}_M\psi$ is both left and right linear. If $M \in Z(\mathbb{K})^{Y\times(X)}$, then ${}_{M^t}\psi = \psi_M$, so that ψ_M is left and right linear. This leads to isomorphisms between the abelian groups $Z(\mathbb{K})^{(X)\times Y} \simeq {}_{(\mathbb{K},\tau)}\mathbf{TopBivect}_{(\mathbb{K},\tau)}(({}_l\mathbb{K}_r^X, \sqcap_X^\tau), ({}_l\mathbb{K}_r^Y, \sqcap_Y^\tau)) \simeq Z(\mathbb{K})^{Y\times(X)}$.

Over a not necessarily commutative ring R, the right notion of an algebra is that of an R-ring (see [1, Definition 10.3, p. 36]), i.e., a ring (with a unit) A together with a (unit-preserving) homomorphism of rings $\eta\colon R \to A$ (this naturally endows A with a structure of a R-R-bimodule: the left-action is given by $r\cdot a = \eta(r)a$ and the right action by $a \cdot r = a\eta(r)$, $a \in A$, $r \in R$). E.g., $\mathbf{TopVect}_{(\mathbb{K},\tau)}(({}_l\mathbb{K}_r^X, \sqcap_X^\tau), ({}_l\mathbb{K}_r^X, \sqcap_X^\tau))$ and, of course, also ${}_{(\mathbb{K},\tau)}\mathbf{TopVect}(({}_l\mathbb{K}_r^X, \sqcap_X^\tau), ({}_l\mathbb{K}_r^X, \sqcap_X^\tau))$ are $\mathbb{K}$-rings. (They are rings under the usual composition of operators, and they are equipped with the respective maps from R, $r \mapsto rid$, and $r \mapsto idr$; the bi-vector space structures thus obtained are the usual ones as it is easily checked.)

For $M \in \mathbb{K}^{Z\times(Y)}$ and $N \in \mathbb{K}^{Y\times(X)}$, one may define a multiplication, which is just the usual matrix multiplication, $MN \in \mathbb{K}^{Z\times(X)}$ as $(MN)(z,x) = \sum_{y\in Y} M(z,y)N(y,x)$ (there are only finitely many terms in this sum because $M \in \mathbb{K}^{Z\times(Y)}$, and $MN \in \mathbb{K}^{Z\times(X)}$ because $N \in \mathbb{K}^{Y\times(X)}$). It is not difficult to check that it is associative, i.e., if $P \in \mathbb{K}^{X\times(W)}$, then $(MN)P = M(NP) \in \mathbb{K}^{Z\times(W)}$, that it admits as a two-sided identity the identity matrix $I_X \in \mathbb{K}^{X\times(X)}$ (given by $I_X(x,x) = 1$, and $I_X(x,x') = 0$ for each $x \neq x'$), and that it is bi-additive. In particular, $\mathbb{K}^{X\times(X)}$ forms a ring under this operation, and even a $\mathbb{K}$-ring with the ring map $\eta\colon \mathbb{K} \to \mathbb{K}^{X\times(X)}$ given by $\eta(r) = rI_X = I_X r$ (once again, the bi-vector space structure induced by η on $\mathbb{K}^{X\times(X)}$ is the usual one).

In a similar way one may define an associative, unital and bi-additive multiplication $\mathbb{K}^{(X)\times Y} \times \mathbb{K}^{(Y)\times Z} \to \mathbb{K}^{(X)\times Z}$. This also gives rise to a $\mathbb{K}$-ring structure on $\mathbb{K}^{(X)\times X}$ (its induced bi-vector space structure is the original one).

Finally, the above multiplications restrict to associative and bi-additive maps $Z(\mathbb{K})^{Z\times(Y)} \times Z(\mathbb{K})^{Y\times(Z)} \to Z(\mathbb{K})^{Z\times(X)}$ and $Z(\mathbb{K})^{(X)\times Y} \times Z(\mathbb{K})^{(Y)\times Z} \to Z(\mathbb{K})^{(X)\times Z}$. In particular, $Z(\mathbb{K})^{(X)\times X}$ and $Z(\mathbb{K})^{X\times(X)}$ are rings. (They are anti-isomorphic one to the other by the transpose map.)

We are now in position to establish a corollary of Theorem 23.

Corollary 2. *For each* $\phi \in \mathbf{TopVect}_{(\mathbb{K},\tau)}(({}_l\mathbb{K}_r^X, \sqcap_X^\tau), ({}_l\mathbb{K}_r^Y, \sqcap_Y^\tau))$ *and* $\psi \in \mathbf{TopVect}_{(\mathbb{K},\tau)}(({}_l\mathbb{K}_r^Y, \sqcap_Y^\tau), ({}_l\mathbb{K}_r^Z, \sqcap_Z^\tau))$, *one has* $M_{\psi\circ\phi} = M_\psi M_\phi$, $M_{id_{\mathbb{K}^X}} = I_X$, *and in particular the* $\mathbb{K}$*-rings* $\mathbf{TopVect}_{(\mathbb{K},\tau)}(({}_l\mathbb{K}_r^X, \sqcap_X^\tau), ({}_l\mathbb{K}_r^X, \sqcap_X^\tau))$ *and* ${}_l\mathbb{K}_r^{X\times(X)}$ *are isomorphic (this means that both the underlying rings and the underlying* $\mathbb{K}$-$\mathbb{K}$*-bi-vector spaces are isomorphic).*

Proof. Let $(z,x) \in Z \times X$. Then,

$$\begin{aligned}
M_{\psi\circ\phi}(z,x) &= {}_l\langle \delta_z, (\psi\circ\phi)(\delta_x)\rangle \\
&= {}_l\langle \delta_z, \psi\left(\sum_{y\in Y}\delta_y\phi(\delta_x)(y)\right)\rangle \\
&= \sum_{y\in Y} {}_l\langle \delta_z, \psi(\delta_y)\rangle \phi(\delta_x)(y) \\
&= \sum_{y\in Y} {}_l\langle \delta_z, \psi(\delta_y)\rangle {}_l\langle \delta_y, \phi(\delta_x)\rangle \\
&= \sum_{y\in Y} M_\psi(z,y)M_\phi(y,x) \\
&= (M_\psi M_\phi)(z,x).
\end{aligned} \tag{29}$$

The remainder of the proof is straightforward. □

Remark 22. The corresponding left version of Corollary 2 is as follows: ${}_{\psi\circ\phi}M = {}_\phi M_\psi M$ for every $\phi \in {}_{(\mathbb{K},\tau)}\mathbf{TopVect}(({}_l\mathbb{K}_r^X, \sqcap_X^\tau), ({}_l\mathbb{K}_r^Y, \sqcap_Y^\tau))$ and $\psi \in {}_{(\mathbb{K},\tau)}\mathbf{TopVect}(({}_l\mathbb{K}_r^Y, \sqcap_Y^\tau), ({}_l\mathbb{K}_r^Z, \sqcap_Z^\tau))$, and, in particular, the $\mathbb{K}$-rings ${}_{(\mathbb{K},\tau)}\mathbf{TopVect}(({}_l\mathbb{K}_r^X, \sqcap_X^\tau), ({}_l\mathbb{K}_r^X, \sqcap_X^\tau))$ and ${}_l\mathbb{K}_r^{(X)\times X}$ are anti-isomorphic (they are isomorphic as bi-vector spaces, however their underlying rings are anti-isomorphic). Finally, the corresponding "two-sided" version states that for every ϕ in ${}_{(\mathbb{K},\tau)}\mathbf{TopBivect}_{(\mathbb{K},\tau)}(({}_l\mathbb{K}_r^X, \sqcap_X^\tau), ({}_l\mathbb{K}_r^Y, \sqcap_Y^\tau))$ and for every ψ in ${}_{(\mathbb{K},\tau)}\mathbf{TopBivect}_{(\mathbb{K},\tau)}(({}_l\mathbb{K}_r^Y, \sqcap_Y^\tau), ({}_l\mathbb{K}_r^Z, \sqcap_Z^\tau))$, $M_{\psi\circ\phi} = M_\psi M_\phi$ and ${}_{\psi\circ\phi}M = {}_\phi M_\psi M$. Moreover the rings ${}_{(\mathbb{K},\tau)}\mathbf{TopBivect}_{(\mathbb{K},\tau)}(({}_l\mathbb{K}_r^X, \sqcap_X^\tau), ({}_l\mathbb{K}_r^X, \sqcap_X^\tau))$ and $Z(\mathbb{K})^{X\times(X)}$ are isomorphic, while they are anti-isomorphic with $Z(\mathbb{K})^{(X)\times X}$ (the transpose map is anti-isomorphism).

Remark 23. Let $\mathbb{K}$ be a division ring and let τ be a Hausdorff division ring topology on $\mathbb{K}$. Let X, Y be two sets. By Theorem 23, if a right linear map $\phi: {}_l\mathbb{K}_r^X \to {}_l\mathbb{K}_r^X$ is continuous for the product topologies $\sqcap_X^\tau$ on $\mathbb{K}^X$ and $\sqcap_Y^\tau$ on $\mathbb{K}^Y$, then it is also continuous with respect to the product topologies $\sqcap_X^\sigma$ and $\sqcap_Y^\sigma$ for any choice of a Hausdorff division ring topology on $\mathbb{K}$. Conversely, if $M \in \mathbb{K}^{Y\times(X)}$, then the right linear map $\psi_M: {}_l\mathbb{K}_r^X \to {}_l\mathbb{K}_r^Y$, of the proof of Theorem 23, is continuous whatever is the Hausdorff division ring topology on $\mathbb{K}$. One observes that, of course, it is possible to define M_ϕ for a right linear map $\phi: {}_l\mathbb{K}_r^X \to {}_l\mathbb{K}_r^Y$ even not continuous. But it is not true that if $M_\phi \in \mathbb{K}^{Y\times(X)}$, then ϕ is continuous. Indeed, let $X = Y$ be assumed infinite. The right vector space spanned by the linearly independent set $\{\,\delta_x : x \in X\,\}$ is $\mathbb{K}_r^{(X)}$. According to the axiom of choice, $\{\,\delta_x : x \in X\,\}$ may be extended to an algebraic right basis B of $\mathbb{K}_r^X$. Let V be the sub-vector space generated by $B \setminus \{\,\delta_x : x \in X\,\} \neq \emptyset$. Of course, $\mathbb{K}_r^X = \mathbb{K}^{(X)} \oplus V$, and let us consider the projection $\pi_V: \mathbb{K}^X \to \mathbb{K}^X$ onto V along $\mathbb{K}^{(X)}$. Since $\mathbb{K}^{(X)} = \ker \pi_V$, for every $x, y \in X$, ${}_l\langle \delta_x, \pi_V(\delta_y)\rangle = 0$, hence M_{π_V} is the zero matrix. Therefore, because $\pi_V \neq 0$, π_V is not continuous (otherwise it would contradict the fact that $M_- : \phi \to M_\phi$ is one-to-one).

7.3. Topological dual and completion

In this section, we explicitly construct the canonical isomorphism between the two-sided topological dual of ${}_l\mathbb{K}_r^{(X)}$ and the two-sided topological dual of ${}_l\mathbb{K}_r^X$, when $\mathbb{K}$ is a complete (Hausdorff) division ring, using the fact that the former is a completion of the later (for some convenient topologies).

Recall the following definition: let R be a ring with a Hausdorff (ring) topology τ, and M be a Hausdorff topological (R,τ)-(R,τ)-bimodule. The *completion* of M is a pair $(\widehat{M}, i)$ where $\widehat{M}$ is complete (Hausdorff) topological (R,τ)-(R,τ)-bimodule (i.e., complete both as a right and as a left topological module) and $i: M \to \widehat{M}$ such that

1. The map i is an isomorphism of topological (R,τ)-(R,τ)-bimodule structures from M into $\widehat{M}$, i.e., i is both an (algebraic) isomorphism and a homeomorphism into $\widehat{M}$ (or in other terms, i is a continuous one-to-one left and right R-linear map, and its inverse $i^{-1}: i(M) \to M$ is continuous where $i(M)$ has the sub-space topology induced by $\widehat{M}$).
2. The image $i(M)$ of M is dense in $\widehat{M}$.
3. For any complete (Hausdorff) (R,τ)-(R,τ)-bimodule N and any continuous and (left and right) linear map $\phi: M \to N$, there exists one, and only one, continuous (left and right) linear map $\widehat{\phi}: \widehat{M} \to N$ such that $\widehat{\phi} \circ i = \phi$.

The notion of a completion for a topological bi-vector space over a topological division ring is similar. In any case the bijection $\phi \mapsto \widehat{\phi}$ provides, for each complete Hausdorff (R,τ)-(R,τ)-bimodule (respectively, bi-vector space) N, an isomorphism of abelian groups from ${}_{(R,\tau)}\mathbf{TopBimod}_{(R,\tau)}(M,N)$ to ${}_{(R,\tau)}\mathbf{TopBimod}_{(R,\tau)}(\widehat{M},N)$ (respectively, from ${}_{(R,\tau)}\mathbf{TopBivect}_{(R,\tau)}(M,N)$ to ${}_{(R,\tau)}\mathbf{TopBivect}_{(R,\tau)}(\widehat{M},N)$).

Lemma 24. *Let (R,τ) be a complete Hausdorff ring. Then, $({}_lR_r^X, \sqcap_X^\tau)$ is the completion of $({}_lR_r^{(X)}, p_X^\tau)$ (the later being equipped with the initial topology p_X^τ with respect to the obvious projections which coincides with the sub-space topology induced by $\sqcap_X^\tau$ on ${}_lR_r^{(X)}$) with $i: {}_lR_r^{(X)} \hookrightarrow {}_lR_r^X$ being the canonical inclusion.*

Proof. It is rather clear that $({}_lR_r^X, \sqcap_X^\tau)$ is complete as a product of complete modules (see for instance [3]). The facts that p_X^τ coincides with the sub-space topology induced by $\sqcap_X^\tau$ on ${}_lR_r^{(X)}$ and that ${}_lR_r^{(X)}$ is dense in ${}_lR_r^X$ are also almost immediate. Let (V,σ) be a (Hausdorff) complete (R,τ)-(R,τ)-bimodule, and let $\phi: ({}_lR_r^{(X)}, p_X^\tau) \to (V,\sigma)$ be a continuous left and right linear map. Let us check that the family $(\phi(\delta_x))_{x\in X}$ is summable in (V,σ). Let U be an open neighborhood of zero in (V,σ). By continuity of ϕ, $\phi^{-1}(U)$ is an open neighborhood of zero. By a well-known characterization of an initial topology by a basis of open sets (see [3]), it follows that there exists a finite subset $X_U \subseteq X$, and for each $x \in X_U$, an open neighborhood U_x of zero in (R,τ) such that $0 \in \bigcap_{x\in X_U} \pi_x^{-1}(U_x) \subseteq \phi^{-1}(U)$. Let $K \subseteq X$ be any finite subset such that $K \cap X_U = \emptyset$. Then, $\pi_x(\delta_y) = \delta_y(x) = 0 \in U_x$ for every $x \in X_U$ and $y \in K$ (since $X_U \cap K = \emptyset$). Hence, by linearity of the projections, $\sum_{y\in K} \delta_y \in \bigcap_{x\in X_U} \pi_x^{-1}(U_x) \subseteq \phi^{-1}(U)$, and thus $\sum_{y\in K} \phi(\delta_y) = \phi\left(\sum_{y\in K} \delta_y\right) \in U$. Since (V,σ) is complete, using Cauchy's condition [21, Definition 10.3 and Theorem 10.4, pp. 63–64], it follows that the family $(\phi(\delta_x))_{x\in X}$ is summable in (V,σ). Then, one may define a map $\widehat{\phi}: {}_lR_r^X \to V$ by $\widehat{\phi}(f) = \sum_{x\in X} f(x)\phi(\delta_x)$. This map is continuous (because it coincides with the unique continuous extension of ϕ to the closure ${}_lR_r^X$ of ${}_lR_r^{(X)}$), and left and right linear. It is of course unique (again by the uniqueness of the extension of a uniformly continuous maps to a closure). □

Let us assume that a ring R has a Hausdorff ring topology τ that makes (R,τ) a complete Hausdorff ring. By Lemma 24, and according to the definition of a completion, taking $({}_lR_r,\tau)$ in place of N, it is clear that there exists a canonical isomorphism Ψ between the two-sided topological dual of $({}_lR_r^{(X)}, p_X^\tau)$ and that of $({}_lR_r^X, \sqcap_X^\tau)$, since for every $\ell \in {}^{\backprime}({}_lR_r^{(X)}, p_X^\tau)'$ there is a unique $\widehat{\ell} \in {}^{\backprime}({}_lR_r^X, \sqcap_X^\tau)'$ such that $\widehat{\ell} \circ i = \ell$ (hence $\Psi(\ell) = \widehat{\ell}$). The isomorphism

$$\begin{array}{rccc} \Psi: & {}^{\backprime}({}_lR_r^{(X)}, p_X^\tau)' & \to & {}^{\backprime}({}_lR_r^X, \sqcap_X^\tau)' \\ & \ell & \mapsto & \widehat{\ell} \end{array} \tag{30}$$

has inverse $\Psi^{-1}(\ell) = \ell \circ i = \ell_{|_{R^{(X)}}}$ for $\ell \in {}^{`}({}_lR^X_r, \sqcap^\tau_X)'$. The isomorphism $\Psi$ may be given an even more explicit description. Let $\ell \in {}^{`}({}_lR^{(X)}_r, p^\tau_X)'$. Then we have

$$\Psi(\ell) = \widehat{\ell}\colon \begin{array}{lcl} R^X & \to & R \\ f & \mapsto & \displaystyle\sum_{x\in X} \ell(\delta_x) f(x). \end{array} \tag{31}$$

Indeed since $f = \sum_{x\in X} \delta_x f(x)$ (sum of a summable family), we have

$$\begin{aligned} \Psi(\ell)(f) &= \widehat{\ell}(f) \\ &= \widehat{\ell}\left(\sum_{x\in X} \delta_x f(x)\right) \\ &= \sum_{x\in X} \widehat{\ell}(\delta_x) f(x) \quad \text{(since } \widehat{\ell} \text{ is continuous and linear)} \\ &= \sum_{x\in X} \ell(\delta_x) f(x) \quad \text{(since } \delta_x \in R^{(X)}\text{)} \end{aligned} \tag{32}$$

Now, one assumes that (R, τ) is actually a complete Hausdorff division ring $(\mathbb{K}, \tau)$. According to previously introduced notations and to the above results, it follows that $\Psi(\ell)(f) = \widehat{\ell}(f) = {}_l\langle p_{\Psi(\ell)}, f\rangle$, where we recall from the proof of Lemma 17, and from Theorem 5, that $p_{\Psi(\ell)} \in Z(\mathbb{K})^{(X)}$ is defined by $p_{\Psi(\ell)}(x) = \Psi(\ell)(\delta_x) = \widehat{\ell}(\delta_x) = \ell(\delta_x)$. Note also that $p_{\Psi(\ell)} = \zeta^{-1}(\Psi(\ell)) = \zeta^{-1}(\widehat{\ell})$.

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