From combinatorial monoids to bialgebras and Hopf algebras, functorially

Laurent Poinsot

Abstract. This contribution provides a study of some combinatorial monoids, namely finite decomposition and locally finite monoids, using some tools from category theory. One corrects the lack of functoriality of the construction of the large algebras of finite decomposition monoids by considering the later as monoid objects of the category of sets with finite-fiber maps. Moreover it is proved that an algebraic monoid (i.e., a commutative bialgebra) may be associated to any finite decomposition monoid, and that locally finite monoids furthermore provide algebraic groups (i.e., commutative Hopf algebras), by attaching in the first case a monoid scheme to the large algebra, and in the second case a group scheme to a subgroup of invertible elements in the large algebra.

1. Introduction

The field of algebraic combinatorics deals with the interface between algebra and combinatorics, i.e., it solves problems from algebra with help of combinatorial methods, and vice versa. The algebras considered are often algebras of "combinatorial" monoids or their deformations. A combinatorial monoid is roughly speaking a usual monoid together with an informal notion of a natural integer-valued size. The size may be the length of elements of a monoid (when this makes sense), but also the number of decompositions of an element into a product of "smaller" pieces. Hence in some sense in a combinatorial monoid the construction or the decomposition of the elements is combinatorially controlled.

Two main classes of such combinatorial monoids have been recognized, namely finite decomposition and locally finite monoids. In a finite decomposition monoid, any element only admits finitely many decompositions into products of two elements. In a locally finite monoid, any element may be decomposed non trivially (i.e., no factors are the identity element) only a finite number of times into a product with finite factors. The first class provides a generalization of both the algebra of a monoid and of the algebra of power series thanks to the notion of a large algebra of a monoid (see [**2, 13**]) which is the set of all functions from the monoid to a given algebra, endowed with a product, called the convolution, inherited from that of the monoid. In other terms the property of finite decomposition allows convolution of functions from the monoid to a given algebra, and relies combinatorial

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objects to algebraic ones. The second class provides a generalization of the famous Möbius inversion formula of number theory [**1**], known for more than a century, and thus an incarnation of one of the main combinatorial principle, namely inclusion-exclusion [**5**]. Moreover this Möbius inversion formula is sometimes (in the case of locally finite posets) also related to the computation of an antipode in some Hopf algebra [**14, 15**], again connecting combinatorial objects to algebraic ones.

The heart of this contribution concerns finite decomposition and locally finite monoids within a category-theoretic setting. We correct the lack of functoriality in the construction of the large algebra of a finite decomposition monoid by considering homomorphisms with finite fibers. This choice is not only adapted but also rather natural since one proves that finite decomposition monoids are precisely the monoid objects in the category of sets with finite-fiber maps (monoidal for the cartesian product), i.e., in some sense the convenient category of combinatorial monoids. We then specialize the construction of the large algebra to the case of locally finite monoids. Local finiteness provides furthermore a group structure on the members of the large algebra that take the value 1 at the identity (this structure being itself related to the Möbius inversion formula). Moreover one also proves that these combinatorial monoids may be used to define (geometrico-algebraic) structures of bialgebras and of Hopf algebras, or algebraic monoids and algebraic groups, because the construction of the large algebra of a finite decomposition monoid actually gives rise to a monoid scheme, i.e., a functor from commutative algebras to monoids, representable as a set-valued functor, and because the group structure on the functions defined on a locally finite monoid that take the value 1 at the identity gives rise to a group scheme, i.e., an Hopf algebra (up to equivalence of categories).

In brief one provides a category-theoretic study of these combinatorial monoids which explains why they play a central role in the interface of algebra and combinatorics. Observe however that this is by no mean a contribution about category theory but it uses some of its tools, which are needed in particular to deal with modern algebraic geometry, and thus in particular to connect monoids to geometric objects.

2. A short introduction to (monoidal) categories

This section is devoted to recall some very basic facts about category theory. With the notion of left adjoint functor (which is only used two times in this contribution but is not recalled hereafter), these are the only concepts to be known for this contribution. The main reference is [**12**].

2.1. About categories. A category $\mathbf{C}$ is given as a class of objects and a class of arrows or morphisms (in $\mathbf{C}$). Any morphism f has a unique *domain* and a unique *codomain* which are objects of $\mathbf{C}$. For each pair of objects (A, B), the set of all morphisms with domain A and codomain B is denoted by $\mathbf{C}(A, B)$. Given two arrows $f \in \mathbf{C}(A, B)$ and $g \in \mathbf{C}(C, D)$, they may be *composed*, or they are *composable*, which means that there is a morphism denoted by $g \circ f \in \mathbf{C}(A, D)$ if, and only if, $B = C$. Given arrows $f \in \mathbf{C}(A, B)$, $g \in \mathbf{C}(B, C)$ and $h \in \mathbf{C}(C, D)$, then $(h \circ g) \circ f = h \circ (g \circ f)$, and of course it belongs to $\mathbf{C}(A, D)$. This means that the composition operation when defined is associative. Finally any object A admits a distinguished arrow $\mathsf{id}_A \in \mathbf{C}(A, A)$ which acts as an identity for the composition: let $f \in \mathbf{C}(A, B)$, then $\mathsf{id}_B \circ f = f = f \circ \mathsf{id}_A$. An *isomorphism* $f \in \mathbf{C}(A, B)$ is a morphism for which there is another morphisms $g \in \mathbf{C}(B, A)$ such that $f \circ g = \mathsf{id}_B$

and $g \circ f = \mathsf{id}_A$. In this case one says that the objects A, B are *isomorphic*, in symbols $A \cong B$.

In this contribution the categories of interests are the following, where R is a commutative ring with a unit,

- **Set**: sets with set-theoretic maps;
- **SemGrp**, **Mon**, **Grp**: the categories of semigroups, monoids and groups with their usual homomorphisms.
- R-**Mod**: the category of R-modules and the usual R-linear maps;
- R-$\mathbf{Alg}_1$, R-**Alg** (resp., R-$\mathbf{CAlg}_1$, R-**CAlg**): the categories of (respectively, commutative) R-algebras, unital or not, and with their usual homomorphisms (that respect the unit when the algebras are unital);
- R-$\mathbf{CompTopAlg}_1$, R-**CompTopAlg**: the categories of complete topological algebras, unital or not, with continuous homomorphisms of algebras (again that are unit-preserving between unital algebras).

One observes that one can define the *product* of two categories $\mathbf{C}, \mathbf{D}$, whose objects are pairs (A, B), A (respectively, B) being an object of $\mathbf{C}$ (respectively, $\mathbf{D}$), and whose morphisms in $\mathbf{C} \times \mathbf{D}((A, B), (C, D))$ are pairs of morphisms $(f, g) \in \mathbf{C}(A, C) \times \mathbf{D}(B, D)$, the composition being given component-wise.

Let $\mathbf{C}, \mathbf{D}$ be two categories. A *functor* $F \colon \mathbf{C} \to \mathbf{D}$ is a pair of maps F that assigns to an object A of $\mathbf{C}$, an object $F(A)$ of $\mathbf{D}$, and to a morphism $f \in \mathbf{C}(A, B)$, a morphism $F(f) \in \mathbf{D}(F(A), F(B))$ such that $F(\mathsf{id}_A) = \mathsf{id}_{F(A)}$ for each object A of $\mathbf{C}$, and $F(g \circ f) = F(g) \circ F(f)$ for a pair of composable morphisms (g, f) in $\mathbf{C}$.

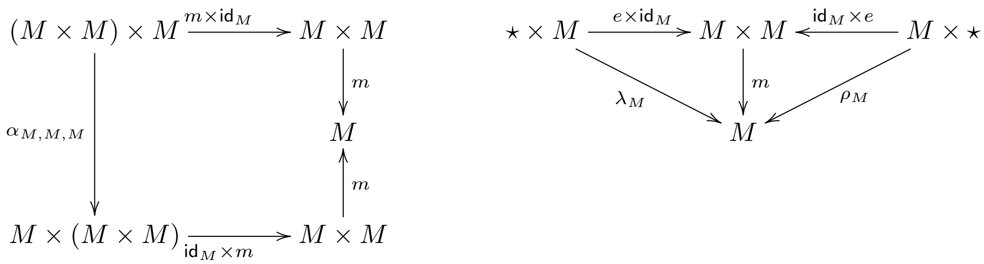
Finally, given two functors $F, G \colon \mathbf{C} \to \mathbf{D}$, a *natural transformation* $\sigma \colon F \Rightarrow G$ from F to G is a family $(\sigma_A)_A$ of morphisms in $\mathbf{D}$ indexed by the objects of $\mathbf{C}$ such that $\sigma_A \in \mathbf{D}(F(A), G(A))$, and for each $\phi \in \mathbf{C}(A, B)$ the following diagram commutes (in $\mathbf{D}$).

$$\begin{array}{ccc} F(A) & \xrightarrow{\sigma_A} & G(A) \\ {\scriptstyle F(\phi)}\downarrow & & \downarrow{\scriptstyle G(\phi)} \\ F(B) & \xrightarrow[\sigma_B]{} & G(B) \end{array} \qquad (2.1)$$

One sometimes says that σ is *natural in* A. Such a natural transformation $(\sigma_A)_A$ with σ_A an isomorphism in $\mathbf{D}$ is referred to as a *natural isomorphism*, and a *natural bijection* when $\mathbf{D} = \mathbf{Set}$.

2.2. About monoidal categories. A monoid $(M, *, 1)$ may be equivalently described as a triple (M, m, e) where M is a set and with two maps $m \colon M \times M \to M$ (given by $m(x, y) = x * y$) and $e \colon \star \to M$ (where $\star$ is any one-element set), namely the constant map with value 1, satisfying the usual associativity, left and right unit laws. Diagrammatically this amounts to the commutativity of the following

diagrams in the category of sets, where $\alpha_{M,M,M}, \lambda_M, \rho_M$ are the obvious bijections.
(2.2)



Forgetting the commutative diagrams on the right and the map e, one characterizes diagrammatically the semigroups. The description of semigroups and monoids in such a way corresponds to the notion of monoid objects in the monoidal category of sets (where the monoidal structure is induced by the cartesian product).

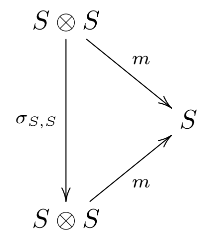
More generally a monoidal category is roughly speaking a convenient setting to develop a theory of monoids. In more details (but not completely detailed, since it is not the main subject of this paper; the reader should refer to [**12**] for the whole description) a monoidal category $\mathbb{C}$ consists of a tuple $(\mathbf{C}, \otimes, I, \alpha, \lambda, \rho)$, where $\mathbf{C}$ is a category, $\otimes\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is a functor, I is an object of $\mathbf{C}$, called its *unit*, α, λ and ρ are natural isomorphisms subject to some axioms called the *coherence axioms*. There are many examples of such categories, e.g., **Set** with the cartesian product, topological spaces with the product topology, R-**Mod** of modules over a commutative ring R with a unit together with the usual tensor product, commutative monoids **CMon** with the tensor product that classifies biadditive maps... Given such a monoidal category $\mathbb{C}$, one may then define its *semigroup* and *monoid objects* which are just characterized as the monoid M above, where now M is an object of $\mathbf{C}$, $m\colon M \otimes M \to M$ and $e\colon I \to M$ are morphisms of $\mathbf{C}$, satisfying the corresponding commutations as in the Diagrams 2.2 (of course, where $\times$ is replaced by $\otimes$). Again semigroup objects are obtained by forgetting the diagrams on the right and the morphism e. For instance, in **Set** with the cartesian product, semigroup and monoid objects are just usual semigroups and monoids, while in the category of modules (respectively, commutative monoids), these are algebras in R-**Alg** and R-$\mathbf{Alg}_1$ (respectively, semi-rings without and with a unit).

These semigroup and monoid objects also come with a definition for their morphisms, so that they form categories. Given two monoid objects $\mathsf{M} = (M, m, e)$ and $\mathsf{N} = (N, n, f)$, a morphism $\phi\colon \mathsf{M} \to \mathsf{N}$ is defined to be a morphism, in $\mathbf{C}$, $\phi\colon M \to N$ such $n \circ (\phi \otimes \phi) = \phi \circ m$ and $\phi \circ e = f$. Again we leave to the reader the verification that the usual notion of homomorphism between monoids is recovered in the set-theoretic case. A morphism between semigroup objects is defined similarly without requiring unit preservation. Hence are formed the categories $\mathbf{SemGrp}(\mathbb{C})$ and $\mathbf{Mon}(\mathbb{C})$ of semigroup and monoid objects in $\mathbb{C}$. For instance, if $\mathbb{C}$ is the monoidal category of R-modules (respectively, commutative monoids), then $\mathbf{Mon}(\mathbb{C})$ and $\mathbf{SemGrp}(\mathbb{C})$ correspond to the categories R-$\mathbf{Alg}_1$ of unital R-algebras and R-**Alg** of not necessarily unital algebras (respectively, semi-rings with or without a unit)

The categorical definition of commutative monoids (or semigroups) requires another ingredient namely a *twist* $\sigma_{A,B}\colon A \otimes B \to B \otimes A$, for each pair (A, B) of objects of $\mathbf{C}$, which again is a natural isomorphism subject to coherence axioms.

A monoidal category with a coherent twist $(\mathbf{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ is referred to as a *symmetric monoidal category.* Now, commutativity of a semigroup (respectively, monoid) object $\mathsf{S} = (S, m)$ (respectively, $\mathsf{S} = (S, m, e)$) in $\mathbb{C}$ is characterized by the following commutative diagram.

(2.3)



The specialization of Diagram 2.3 to the case of the category of sets obviously gives the usual axiom of commutativity. For sets one has of course $\sigma_{A,B}(a, b) = (b, a)$, $(a, b) \in A \times B$, while for R-**Mod** it is $\sigma_{A,B}(a \otimes b) = b \otimes a$. The category of all commutative semigroup (respectively, monoid) objects, with the same morphisms as non-necessarily commutative objects, is denoted by $\mathbf{CSemGrp}(\mathbb{C})$ (respectively, $\mathbf{CMon}(\mathbb{C})$).

For instance, if $\mathbb{C}$ is the category R-**Mod** with its usual tensor $\otimes_R$, then one recovers the categories R-**CAlg** and R-$\mathbf{CAlg}_1$ of commutative R-algebras without or with a unit. In what follows, one denotes by $\mathsf{A} = (A, m, e)$ a commutative R-algebra with a unit seen as an object of the category of commutative monoid objects in the symmetric monoidal category of modules over R, where $m \colon A \otimes_R A \to A$ and $e \colon R \to A$ are respectively its multiplication and its unit. In particular, here A denotes the underlying R-module of the algebra A. One recovers the classical definition of an algebra $\mathsf{A} = (A, \cdot, 1_{\mathsf{A}})$ by considering $x \cdot y = m(x \otimes y)$ and $1_{\mathsf{A}} = e(1_R)$. One denotes by $|\mathsf{A}|$ the underlying set of the algebra A.

3. Combinatorial monoids

The heart of this contribution concerns two classes of combinatorial monoids, namely finite decomposition and locally finite monoids. They have recently knew a regain of interests [**9, 10**] in algebraic combinatorics, but they are known from a long time (see [**2, 5, 11**]). In the Subsection 3.1, one shows that finite decomposition monoids are the monoid objects of the category of sets with finite-fiber maps. This is then used to provide a functorial relation between them and their large algebras. In the Subsection 3.3 one proves in particular that local finiteness provides a structure of a group under convolution to the members of the large algebra that takes the value 1 at the identity of the monoid.

3.1. Finite decomposition. By "combinatorial" monoid (semigroup) is meant two particular classes of monoids (semigroups) extremely useful in the domain of algebraic combinatorics, namely *finite decomposition* and *locally finite monoids* (*semigroups*).

Let $(S, *)$ be a semigroup. It is said to be a *finite decomposition semigroup* whenever for each $x \in S$, there are only finitely many $(y, z) \in S^2$ such that $x = y * z$. A *finite decomposition monoid* is a monoid $(M, *, 1)$ whose underlying semigroup $(M, *)$ is a finite decomposition semigroup. Bourbaki in [**2**] says that such a semigroup or monoid satisfies the *property* (D). Any finite semigroup (monoid) is of course a finite decomposition semigroup (monoid).

Let us define the category **FinFib** of all sets with *finite-fiber maps* between them, i.e., a map $f\colon X \to Y$ such that for each $y \in Y$, $f^{-1}(\{y\})$ is a finite subset of X. It is clear that the (usual) composition of two such maps remains a finite-fiber map, and that any bijection is a finite-fiber map, so are all identity maps, hence **FinFib** is indeed a category, and even a (non-full) subcategory of sets. This category admits a monoidal structure. Namely, $\mathbb{FinFib} := (\mathbf{FinFib}, \times, \star, \alpha, \lambda, \rho, \sigma)$ is a symmetric monoidal category, where $\times$ is the usual cartesian product of sets, $\star$ is a chosen one-element set, $\alpha, \lambda, \rho, \sigma$ are the coherence isomorphisms of the symmetric monoidal category of sets (since these are isomorphisms they belong to **FinFib**). Coherence axioms are of course satisfied here since they are satisfied for sets. One observes that $\times$ is no more a *categorical product* (see [**12**]) in **FinFib** since the canonical projections associated to a product of sets are not necessarily finite-fiber maps. The following result is rather easy since the property of finite decomposition for a monoid means that the multiplication $*$ has finite fibers (moreover the unit of a monoid, seen as a map $1\colon \star \to M$, is of course always a finite-fiber map).

PROPOSITION 3.1. *Finite decomposition semigroups (monoids) are precisely the semigroup (monoid) objects in* $\mathbb{FinFib}$. *Moreover the adjunction of a unit to a finite decomposition semigroup yields to a finite decomposition monoid*[1].

One observes the class of finite decomposition monoids (respectively, semigroups) is closed under sub-object.

REMARK 3.2. The category $\mathbf{SemGrp}(\mathbb{FinFib})$ (respectively, $\mathbf{Mon}(\mathbb{FinFib})$) has objects the finite decomposition semigroups (respectively, monoids). It is of course a subcategory of that of semigroups (respectively, monoids). One observes that it is not the (full sub)category of semigroups (respectively, monoids) which have the finite decomposition property because the morphisms are different (in $\mathbf{SemGrp}(\mathbb{FinFib})$, respectively, $\mathbf{Mon}(\mathbb{FinFib})$, the morphisms are usual homomorphisms of semigroups, respectively, monoids, but they are also required to have finite fibers; in what follows we refer to them as *morphisms* of finite decomposition semigroups and monoids), but it is a subcategory of **SemGrp** (respectively, **Mon**).

One of the main motivation for considering finite decomposition semigroups (monoids) is that they allow convolution of functions. More precisely, let $\mathsf{S} = (S, *)$ be a such a semigroup (or a monoid $\mathsf{S} = (S, *, 1)$). Let R be any commutative ring with a unit, and let $\mathsf{A} = (A, \cdot, 1_{\mathsf{A}})$ be a commutative R-algebra with a unit. Then, the set $|\mathsf{A}|^S$ of all maps from S to the carrier set $|\mathsf{A}|$ of the algebra A admits a structure of a R-algebra[2] (commutative if, and only if, S is commutative) given as follows (this multiplication is called the *convolution*)

$$(f \star g)(x) = \sum_{y * z = x} f(y) \cdot g(z)$$

for all $f, g \in |\mathsf{A}|^S$ and $x \in S$. This algebra is denoted by $\mathsf{A}_R[[\mathsf{S}]]$ and is called the *large algebra* (or just the *large ring* if $R = \mathbb{Z}$) of S (see [**2**]). S embeds into (the underlying multiplicative semigroup or monoid of) this algebra by the homomorphism

[1] In category-theoretic terms, this means that the forgetful functors from the category of finite decomposition monoids to that of finite decomposition semigroups, which forgets the unit, has a left adjoint.

[2] Its additive structure being defined point-wise using that of the underlying R-module A of the algebra A.

$x \mapsto \delta_x$ which takes the value 1_{A} only for x and 0 otherwise. In case S is a monoid with a unit 1, then this R-algebra also is unital (with unit the map δ_1 which one simply denotes by 1 without risk of confusion since M embeds into $\mathsf{A}_R[[\mathsf{M}]]$).

REMARK 3.3. The same holds if one replaces R by the semi-ring $\mathbb{N}$. We thus obtain a *large semi-ring* (with or without a unit) $\mathbb{N}[[\mathsf{S}]]$.

EXAMPLE 3.4. In case of the free commutative monoid $\mathbb{N}^{(X)}$ on X, $\mathsf{A}_R[[\mathbb{N}^{(X)}]]$ is just the usual algebra of formal power series in the commutative indeterminates in X (in particular, one has $\mathsf{A}_R[[\mathbb{N}]] \cong A[[x]]$), while in case of the free monoid $X^\star$ on a set X, $\mathsf{A}_R[[X^\star]]$ is the ring of non-commutative formal power series $\mathsf{A}\langle\langle X \rangle\rangle$ (see [**16**]) with coefficients in A, seen as a R-algebra.

Let $f\colon X \to Y$ be a finite-fiber map. Let $\mathsf{A} = (A, m, e)$ be a commutative R-algebra with a unit (where again R is a commutative ring with a unit). Then, one can define $|\mathsf{A}|^f\colon |\mathsf{A}|^X \to |\mathsf{A}|^Y$ by

$$(|\mathsf{A}|^f(g))(y) = \sum_{x \in f^{-1}(\{\, y \,\})} g(x)$$

for any $g \in |\mathsf{A}|^X$, and $y \in Y$. One observes that the sum has only finitely many non-zero terms, and thus is a usual algebraic sum. This definition of course requires the fact that f has only finite fibers, and does not hold for any choice of the set-theoretic map f. Moreover, the R-module structure of A induces an obvious structure of R-module on $|\mathsf{A}|^X$, which is denoted by A^X. Of course for any finite-fiber map $f\colon X \to Y$, $|\mathsf{A}|^f$ is a R-linear map from A^X to A^Y denoted by $A^f\colon A^X \to A^Y$. Any R-module of the form A^X has a somewhat natural topology (for which it becomes a topological R-module [**18**], i.e., the addition $+\colon A^X \times A^X \to A^X$, the additive inverse $-\colon A^X \to A^X$, and the scalar multiplication $R \times A^X \mapsto A^X$ are continuous, where on a cartesian product is considered the product topology, with R and A being discrete), namely the product topology on $|\mathsf{A}|^X$ with $|\mathsf{A}|$ and R being equipped with the discrete topology. If M is another topological R-module (again with R discrete), and if $f\colon M \to A^X$ is a linear map, then f is continuous if, and only if, for each $x \in X$, the R-linear map $m \in M \mapsto (f(m))(x) \in A$ is continuous. Equivalently it is the coarsest module topology on A^X that makes continuous the projections $\pi_x\colon g \in A^X \mapsto g(x) \in A$, $x \in X$. Given $f\colon X \to Y$, we may check that $A^f\colon A^X \to A^Y$ is continuous since $g \mapsto (A^f(g))(y)$ is just a finite sum of projections. Of course for the chosen topology A^X is a complete R-module (and even the completion of the topological, and Hausdorff, module $A^{(X)}$ of finitely-supported maps from X to A, i.e., those maps with only finitely many non-zero values, equipped with the coarsest topology making continuous each map $f \in A^{(X)} \mapsto f(x) \in A$, with A and R discrete). The following result is thus obtained.

PROPOSITION 3.5. *Let $\mathsf{A} = (A, \cdot, 1_{\mathsf{A}})$ be a commutative R-algebra with a unit. Then $A^-\colon X \mapsto A^X$ and $A^-\colon f \mapsto A^f$, where f is a finite-fiber map, actually defines a functor $A^-\colon$ **FibFin** $\to R$-**CompTopMod**, where R-**CompTopMod** is the category of complete (hence Hausdorff) topological modules over a discrete R.*

Moreover the functor from Proposition 3.5 may be lifted to the categories **SemGrp**($\mathbb{FinFib}$) and **Mon**($\mathbb{FinFib}$). By this is meant that $A^\phi\colon A^M \to A^N$ is actually a homomorphism of topological R-algebras, denoted by $\mathsf{A}_R[[\phi]]$, from $\mathsf{A}_R[[\mathsf{M}]]$

to $\mathsf{A}_R[[\mathsf{N}]]$, whenever $\phi\colon \mathsf{M} \to \mathsf{N}$ is a morphism of finite decomposition semigroups (or monoids, and in this case A^{ϕ} is unit-preserving). Of course for the product topology, $\mathsf{A}_R[[\mathsf{M}]]$ is complete (with both A and R with the discrete topology). Hence we get

Proposition 3.6. *Let* $\mathsf{A} = (A, \cdot, 1_{\mathsf{A}})$ *be a commutative* R*-algebra with a unit. Then, both correspondences* $A_R[[-]]\colon \mathbf{SemGrp}(\mathbb{FibFin}) \to R\text{-}\mathbf{CompTopAlg}$ *and* $\mathsf{A}_R[[-]]\colon \mathbf{Mon}(\mathbb{FibFin}) \to R\text{-}\mathbf{CompTopAlg}_1$ *define functors.*

Let us denote by S^1 the (finite decomposition) monoid obtained from the (finite decomposition) semigroup by *unitarization*, i.e., by a free adjunction of an identity element, and let A^1 be the same for a non-unital topological algebra[3], then of course $\mathsf{A}_R[[\mathsf{S}^1]] \cong \mathsf{A}_R[[\mathsf{S}]]^1$ (where $\cong$ is the isomorphism relation in the category of topological R-algebras, R discrete). This is the content of the following result.

Proposition 3.7. *The large algebra of the unitarization of a finite decomposition semigroup is isomorphic (both algebraically and topologically) to the unitarization of the large algebra of the same finite decomposition semigroup. In particular the large algebra of a finite decomposition semigroup may be identified with a two-sided ideal of the large algebra of its unitarization.*

It is also quite easy to prove that $\mathsf{A}_R[[\mathsf{M}]]$ is just the completion of the usual monoid R-algebra $\mathsf{A}[\mathsf{M}]$ under its coarsest topology that makes continuous the map $f \in \mathsf{A}[\mathsf{M}] \mapsto f(x) \in \mathsf{A}$ for each $x \in M$.

3.2. Filtered monoids. Let X be any set, and let $(X_n)_{n\in\mathbb{N}}$ be a family of subsets of X such that $X_m \subseteq X_n$ whenever $m \geq n$. Then, $(X_n)_n$ is referred to as a (decreasing) *filtration* and $(X, (X_n)_n)$ is said to be a *filtered set*. If it happens that $X_0 = X$, then $(X, (X_n)_n)$ is an *exhausted filtered set*. Let $(X, (X_n)_n)$, $(Y, (Y_n)_n)$ be two filtered sets. Let $f\colon X \to Y$ be a finite-fiber map. Then, it is said to be a *finite-fiber filtration-respecting map* if for each n, $f(X_n) \subseteq Y_n$. Of course composing such maps gives such a map, and the identity on each set may be seen in a straightforward way as a finite-fiber filtration-respecting map. Hence obtains a category, $\mathbf{FiltFinFib}_0$ (respectively, $\mathbf{FiltFinFib}$), namely the category of filtered sets (respectively, exhausted filtered sets) with finite-fiber filtration-respecting maps.

Let again $(X, (X_n)_n)$, $(Y, (Y_n)_n)$ be two filtered sets, and let us define a new one by $(X, (X_n)_n) \otimes (Y, (Y_n)_n)$ by $(X \times Y, (T^n(X, Y))_n)$, where $T^n(X, Y) = \bigcup_{i=0}^{n} X_i \times Y_{n-i}$. When the two filtered sets are exhausted, it is again the case for this one (since $T^0(X, Y) = X_0 \times Y_0 = X \times Y$). Let $\star$ be any one-element set, and define $\mathbf{1}_n$ as the empty set whenever $n > 0$, and $\mathbf{1}_0 = \star$. Thus, $(\star, (\mathbf{1}_n)_n)$ is a (exhausted) filtered set, and it satisfies $(X, (X_n)_n) \otimes (\star, (\mathbf{1}_n)_n) \cong (X, (X_n)_n) \cong (\star, (\mathbf{1}_n)_n) \otimes (X, (X_n)_n)$ in the category $\mathbf{FiltFinFib}_0$ (and also in $\mathbf{FiltFinFib}$ if $(X, (X_n)_n)$ is exhausted). For every (exhausted) filtered sets $(X, (X_n)_n)$, $(Y, (Y_n)_n)$ and $(Z, (Z_n)_n)$, the coherence maps $\alpha_{X,Y,Z}, \lambda_X, \rho_X$ and $\sigma_{X,Y}$ of the symmetric monoidal category of sets with the cartesian product give rise to finite-fiber filtration-preserving maps between the corresponding (exhausted) filtered sets. From that it follows directly that $\mathbf{FiltFinFib}_0$ (respectively, $\mathbf{FiltFinFib}$) becomes a symmetric monoidal category with $\otimes$ denoted by $\mathbb{FiltFinFib}_0$ (respectively, $\mathbb{FiltFinFib}$).

[3] The construction of A^1 from A is just the algebraic unitarization of the algebra A, i.e., as an algebra $\mathsf{A}^1 \cong \mathsf{A} \times R$, equipped with the product topology, R being discrete.

So what are the semigroup and monoid objects in these monoidal categories? Let $((M,(M_n)_n),m,e)$ be a monoid object in $\mathbb{FiltFinFib}_0$. Then, in particular, with $m\colon M\times M\to M$ and $e\colon \star\to M_0\subseteq M$, (M,m,e) is a finite decomposition monoid, and (M_0,m_0,e), where m_0 is the restriction of m to $M_0\times M_0$ (since $m(M_0\times M_0)\subseteq M_0$), is submonoid of (M,m,e). Because for each n, $m(\bigcup_{i=0}^{n} M_i\times M_{n-i})\subseteq M_n$, one also has $m(M_m\times M_n)\subseteq M_{m+n}$ for each m,n. In particular any member M_n of the filtration is a two-sided ideal of the monoid (M_0,m_0,e). In a similar way a monoid object $((M,(M_n)_n),m,e)$ in $\mathbb{FiltFinFib}$ gives rise to a usual finite decomposition monoid (M,m,e) (here $M=M_0$), hence the M_n's are two-sided ideals of M, and $\mathbf{Mon}(\mathbb{FiltFinFib})$ is a full subcategory of $\mathbf{Mon}(\mathbb{FiltFinFib}_0)$. Each category $\mathbf{Mon}(\mathbb{FiltFinFib}_0)$ and $\mathbf{Mon}(\mathbb{FiltFinFib})$ thus has a forgetful functor into the category $\mathbf{Mon}(\mathbb{FinFib})$ given by $((M,(M_n)_n),m,e)\mapsto (M,m,e)$, and there is a functor $((M,(M_n)_n),m,e)\mapsto ((M_0,(M_n)_n),m_0,e)$ from $\mathbf{Mon}(\mathbb{FiltFinFib}_0)$ to $\mathbf{Mon}(\mathbb{FiltFinFib})$ which may be proved to be a right adjoint to the natural inclusion[4] functor $\mathbf{Mon}(\mathbb{FiltFinFib})\hookrightarrow\mathbf{Mon}(\mathbb{FiltFinFib}_0)$. Semigroup objects have also the similar property.

3.3. Local finiteness. Another important class of "combinatorial" semigroups or monoids is that of locally finite semigroups or monoids. Let $\mathsf{S}=(S,*)$ be a semigroup, and let $x\in S$. Then, $D_n(x)=\{\,(x_1,\cdots,x_n)\in S^n\colon x=x_1*\cdots*x_n,\,\}$ is the set of all *decompositions* of length n of x. The semigroup S is said to be *locally finite* (see [**5, 11**]) whenever each $x\in S$ only admits finitely many different decompositions, i.e., $\bigcup_{n\geq 0} D_n(x)$ is a finite set.

Of course any locally finite semigroup is a finite decomposition semigroup. But observe that not all finite decomposition semigroups are also locally finite (consider any non-trivial finite group).

Now, let $\mathsf{M}=(M,*,1)$ be a monoid. A *non-trivial decomposition* (of length n) of $x\in M$ is a finite sequence $(x_1,\cdots,x_n)\in(M\setminus\{\,1\,\})^n$ such that $x_1*\cdots*x_n=x$. The set of all non-trivial decompositions of length n of x is denoted by $D_n^*(x)$. A monoid M is said to be *locally finite* whenever for each x, the set $\bigcup_{n\geq 0} D_n^*(x)$ is a finite set. The following result is clear (see [**5**]).

PROPOSITION 3.8. *A monoid* M *is locally finite if, and only if, it is a finite decomposition monoid, and there exists an integer* n *such that* $D_m^*(x)=\emptyset$ *for all* $m>n$.

EXAMPLE 3.9. There are infinite locally finite semigroups or monoids, e.g. any free, or free commutative or even free partially-commutative (see [**5**]), semigroup or monoid. More generally any quotient of a free semigroup (or monoid) by a multi-homogeneous congruence ([**8**]), i.e., a congruence that preserves the number of each variable in a word, is of course a locally finite semigroup (or monoid).

It is easy to check that if $\mathsf{M}=(M,*,1)$ is a locally finite monoid, then $M^*=M\setminus\{\,1\,\}$ is a locally finite semigroup (with the restriction of the multiplication $*$), i.e., the only invertible element of M is the unit 1. Following [**6**] one says more generally that a monoid M is *conical* whenever $(M^*,*)$ is a semigroup (one calls it the semigroup of *non-identity elements* of M and denotes it by M^*). Equivalently this means that a conical monoid is just (isomorphic to) the unitarization S^1 of

[4]Following [**12**] one says that $\mathbf{Mon}(\mathbb{FiltFinFib})$ is a coreflective subcategory of $\mathbf{Mon}(\mathbb{FiltFinFib}_0)$.

some semigroup S (because $(\mathsf{S}^1)^* \cong \mathsf{S}$). Hence, one can alternatively define a locally finite monoid to be a conical monoid M with a locally finite semigroup of non-identity elements M^*. Moreover, given locally finite monoids M and N, and a morphism $\phi\colon \mathsf{M} \to \mathsf{N}$ (hence a finite-fiber homomorphism), then $\phi^{-1}(\{1\}) = \{1\}$ (indeed, if there exists $x \neq 1$ such that $\phi(x) = 1$, then for every integer $n > 0$, $\phi(x^n) = 1$, but $x^n \neq x^m$ for each $n \neq m$, otherwise assuming e.g. $n > m$, one has $x^{m+(n-m)} = x^m$, and then $x^m = x^m x^{p(n-m)} = x^m \underbrace{x \cdots x}_{p(n-m)\ factors}$ for each p implying that x^m admits arbitrary long non-trivial decomposition in contradiction with local finiteness, thus $\phi^{-1}(\{1\})$ has infinitely many members contradicting its finite-fiber property). Hence, the correspondence $\mathsf{M} \mapsto \mathsf{M}^*$ is functorial, and $(\mathsf{M}^*)^1 \cong \mathsf{M}$. The following result becomes obvious.

Proposition 3.10. *The categories of locally finite monoids and of locally finite semigroups, seen respectively as (full) subcategories of* $\mathbf{Mon}(\mathbb{FinFib})$ *and of* $\mathbf{Sem}(\mathbb{FinFib})$*, are isomorphic.*

If $\mathsf{M} = (M, *, 1)$ is a locally finite monoid, then one can define a *length function* $\ell\colon M \to \mathbb{N}$ as $\ell(x) = \max\{n \in \mathbb{N}\colon D^*_n(x) \neq \emptyset\}$ (of course, $\ell(1) = 0$). One easily checks that $\ell(x*y) \geq \ell(x) + \ell(y)$ and $\ell(x) = 0$ if, and only if, $x = 1$. For each $n \in \mathbb{N}$, let us define $M_n = \{x \in M\colon \ell(x) \geq n\}$. It is clear that $M_0 = M$, and $M_m \subseteq M_n$ for each $m \geq n$. Moreover, $1 \in M_0$, and for each n, for each $i \in \{0, \cdots, n\}$, and for each $(x, y) \in M_i \times M_{n-i}$, $\ell(x*y) \geq \ell(x) + \ell(y) \geq n$, so that $x*y \in M_n$. This shows that $((M, (M_n)_n), *, e)$, with $e\colon \star \to M$ the constant map with value 1, is a monoid object in $\mathbb{FiltFinFib}$. Of course a similar result holds for locally finite semigroups.

Let $\mathsf{M} = (M, *, 1)$ be a locally finite monoid. Let R be a commutative ring with a unit, and let $\mathsf{A} = (A, \cdot, 1_{\mathsf{A}})$ be a commutative R-algebra with a unit. Then, for any locally finite monoid M one may of course form the large algebra $\mathsf{A}_R[[\mathsf{M}]]$. But now, one can define another topology on $\mathsf{A}_R[[\mathsf{M}]]$, which is finer than the product topology, as follows. The length function ℓ of M extends to an *order function* on $\mathsf{A}_R[[\mathsf{M}]]$ by

$$v(f) = \inf\{n\colon \exists x \in M_n,\ f(x) \neq 0\}$$

for $f \in \mathsf{A}_R[[\mathsf{M}]]$, the infimum being taken in the poset $\mathbb{N} \cup \{\infty\}$ obtained from $\mathbb{N}$ by adjunction of the greatest lower bound ∞. In particular, $v(f) = \infty$ if, and only if, $f = 0$. The following holds:

(1) $v(1) = 0$;
(2) $v(f - g) \geq \min\{v(f), v(g)\}$;
(3) $v(f \star g) \geq v(f) + v(g)$.

Let us also consider the augmentation ideal $\mathfrak{M}(\mathsf{A}, \mathsf{M})$ of the large algebra $\mathsf{A}_R[[\mathsf{M}]]$, namely the set of all maps from M to $|\mathsf{A}|$ vanishing at 1, i.e., $\{f \in \mathsf{A}_R[[\mathsf{M}]]\colon v(f) \geq 1\}$. Finally, for each $n \in \mathbb{N}$, let $\mathfrak{M}(\mathsf{A}, \mathsf{M})_{\geq n} = \{f \in \mathsf{A}_R[[\mathsf{M}]]\colon v(f) \geq n\}$ (in particular, $\mathfrak{M}(\mathsf{A}, \mathsf{M})_0 = \mathsf{A}_R[[\mathsf{M}]]$ and $\mathfrak{M}(\mathsf{A}, \mathsf{M})_1 = \mathfrak{M}(\mathsf{A}, \mathsf{M})$). Then, $(\mathfrak{M}(\mathsf{A}, \mathsf{M})_{\geq n})_{n \in \mathbb{N}}$ is a decreasing (separated and exhaustive) filtration in the usual meaning of commutative algebra [**3**], and thus forms a basis of neighborhoods of zero, for the algebra $\mathsf{A}_R[[\mathsf{M}]]$, so that it becomes an Hausdorff topological R-algebra (with A and R assumed discrete), the separation comes from $\bigcap_{n \geq 0} \mathfrak{M}(\mathsf{A}, \mathsf{M})_{\geq n} = (0)$. This topology is always finer than the product topology, and may be even strictly finer, for instance when there are infinitely many members of M of a given length (in this case, let $(x_n)_{n \in \mathbb{N}}$ be an infinite sequence of pairwise distinct members of M of length say

ℓ, then $f_n = \sum_{k=0}^{n} x_k$ converges with respect to the product topology but not in the topology defined by the decreasing filtration $(\mathfrak{M}(\mathsf{A},\mathsf{M})_{\geq n})_n$ because $v(f_n) = \ell$ for each n), while they coincide when for each n, there are only finitely many $x \in M$ such that $\ell(x) = n$. It is not difficult to see that $\mathsf{A}_R[[\mathsf{M}]]$ is also complete for the topology defined by the decreasing filtration.

REMARK 3.11. Although $\mathsf{A}_R[[\mathsf{M}]]$ is complete for the topology defined by the filtration, it is not in general the completion of the monoid algebra $\mathsf{A}[\mathsf{M}]$ under the topology defined by the induced filtration $(\mathsf{A}[\mathsf{M}] \cap \mathfrak{M}(\mathsf{A},\mathsf{M})_{\geq n})_n$. Let us assume that there exists some ℓ such that there are infinitely many members of M of length ℓ. Let $(x_n)_{n\in\mathbb{N}}$ be an infinite sequence of pairwise distinct members of M of length say ℓ, then $f_n = \sum_{k=0}^{n} x_k$, while being a Cauchy sequence, does not admits a limit in the completion of $\mathsf{A}[\mathsf{M}]$ under its induced filtration. But of course it has a limit (in the product topology) in $\mathsf{A}_R[[\mathsf{M}]]$.

LEMMA 3.12. *For each $f \in \mathfrak{M}(\mathsf{A},\mathsf{M})$, $1-f$ is invertible, and $1-f = \sum_{n\geq 0} f^n$.*

PROOF. Since $v(f^n) \to \infty$ when $n \to \infty$, it follows that the family $(f^n)_n$ is topologically nilpotent, i.e., it is summable, with sum $\sum_{n\geq 0} f^n$ (see [**3, 18**]). Let $n \in \mathbb{N}$, $(1-f)\sum_{k=0}^{n} f^k = 1 - f^{n+1} \to 1$ with n. Because $\mathsf{A}_R[[\mathsf{M}]]$ is a topological algebra, the convolution is jointly and thus also seperately continuous, so that $(1-f)\sum_{n\geq 0} f^n = 1$. □

According to Lemma 3.12, one may define the *star operation*, well-known in language theory, by $f \in \mathfrak{M}(\mathsf{A},\mathsf{M}) \to f^\star = \sum_{n\geq 0} f^n \in \mathsf{U}(\mathsf{A}_R[[\mathsf{M}]])$ (where $\mathsf{U}(\mathsf{A})$ denotes the group of multiplicatively invertible elements of an algebra A).

Let us consider the set $1 + \mathfrak{M}(\mathsf{A},\mathsf{M}) = \{\, f\colon M \to |\mathsf{A}| \colon f(1) = 1_{\mathsf{A}} \,\}$. Of course, any member f of $1 + \mathfrak{M}(\mathsf{A},\mathsf{M})$ may be written uniquely as a sum $f = 1 + g$ for $g \in \mathfrak{M}(\mathsf{A},\mathsf{M})$ given by $g(1) = 0$, and $g(x) = f(x)$ for each $x \in M$.

LEMMA 3.13. *The set $1 + \mathfrak{M}(\mathsf{A},\mathsf{M})$ is a subgroup of $\mathsf{U}(\mathsf{A}_R[[\mathsf{M}]])$.*

PROOF. It is clearly closed under convolution. One only needs to prove that $f^\star \in 1 + \mathfrak{M}(\mathsf{A},\mathsf{M})$ for each $f \in \mathfrak{M}(\mathsf{A},\mathsf{M})$. For each $n > 0$, $f^n(1) = 0$. Since the projection $\pi_1 \colon \mathsf{A}_R[[\mathsf{M}]] \to \mathsf{A}$ is continuous (for the product topology), $f^\star(1) = \pi_1(f^\star) = \pi_1(1 + \sum_{n\geq 1} f^n) = 1 + \sum_{n\geq 1} \pi_1(f^n) = 1$. □

The group structure on $1 + \mathfrak{M}(\mathsf{A},\mathsf{M})$ provides a generalization of the classical Möbius inversion formula.

PROPOSITION 3.14. *(Möbius inversion formula.) Let $\zeta \in 1 + \mathfrak{M}(\mathsf{A},\mathsf{M})$ be defined by $\zeta(x) = 1_A$ for every $x \in M$. Let μ be the inverse of ζ (whose existence follows from Lemma 3.13). Let $f, g \in \mathsf{A}_R[[\mathsf{M}]]$. Then the following assertions are equivalent:*

- *For all $x \in M$, $g(x) = \displaystyle\sum_{y*z=x} f(y)$.*
- *For all $x \in M$, $f(x) = \displaystyle\sum_{y*z=x} g(y)\mu(z)$.*

PROOF. This is obvious since the first assertion means that $g = f \star \zeta$, and the second is equivalent to $f = g \star \zeta^{-1} = g \star \mu$. □

Let us also add some functorial observations. Until the end of this subsection, one considers a locally finite monoid $(M, *, 1)$ as the monoid object $((M, (M_n)_n), *, e)$ in $\mathbb{FiltFinFib}$. In particular the morphisms are required to be finite-fiber and filtration-preserving maps. Let $\phi\colon \mathsf{M} = (M, *, 1) \to \mathsf{N} = (N, *, 1)$ be a morphism between locally finite monoids. Let $y \in N$. Let $x \in \phi^{-1}(\{ y \})$. If $\ell(x) > \ell(y)$, then $\ell(y) = \ell(\phi(x)) \geq \ell(x) > \ell(y)$ (because $\phi(M_{\ell(x)}) \subseteq N_{\ell(x)}$) which is a contradiction. Thus, for all y, for all $x \in M$ such that $\phi(x) = y$, one has $\ell(y) \geq \ell(x)$. Now, let A be a commutative R-algebra with a unit, and let $g \in \mathfrak{M}(\mathsf{A}, \mathsf{M})_{\geq n}$. Let $y \in N$ with $\ell(y) < n$. Then, it follows that $(\mathsf{A}_R[[\phi]](g))(y) = \sum_{x \in \phi^{-1}(\{ y \}} g(x) = 0$ so that $\mathsf{A}_R[[\phi]](g) \in \mathfrak{M}(\mathsf{A}, \mathsf{M})_{\geq n}$, hence $\mathsf{A}_R[[\phi]](\mathfrak{M}(\mathsf{A}, \mathsf{M})_{\geq n}) \subseteq \mathfrak{M}(\mathsf{A}, \mathsf{M})_{\geq n}$ for each n which shows that $\mathsf{A}_R[[\phi]]$ is a continuous map (with respect to the filtrations).

Therefore the assignment $\mathsf{M} \mapsto \mathsf{A}_R[[\mathsf{M}]]$ provides a functor from the category of locally finite monoids (with finite-fiber and filtration-preserving homomorphisms) to that of complete topological algebras over R.

This functor restricts to a functor from the category of locally finite monoids (with finite-fiber and filtration-preserving homomorphisms) to the category of groups under the assignment $\mathsf{M} \mapsto 1 + \mathfrak{M}(\mathsf{A}, \mathsf{M})$. To see this we just need to check that $\mathsf{A}_R[[\phi]](1+f) \in 1+\mathfrak{M}(\mathsf{A}, \mathsf{N})$ for every morphism $\phi\colon \mathsf{M} \to \mathsf{N}$ and every $f \in \mathfrak{M}(\mathsf{A}, \mathsf{M})$ (because every homomorphism of algebras associates an invertible element to an invertible element). This is clear since $\mathsf{A}_R[[\phi]](1 + f) = 1 + \mathsf{A}_R[[\phi]](f)$, and $(\mathsf{A}_R[[\phi]](f))(1) = \sum_{x \in \phi^{-1}(\{ 1 \})} f(x) = f(1) = 0$ (because $\phi^{-1}(\{ 1 \}) = \{ 1 \}$ for every finite-fiber homomorphisms between locally finite monoids). Hence denoting by **LFMon** the category of locally finite monoids with finite-fiber and filtration-preserving maps, one has

PROPOSITION 3.15. *For every commutative R-algebra with a unit, each assignment $\mathsf{M} \mapsto \mathsf{A}_R[[\mathsf{M}]]$ and $\mathsf{M} \mapsto 1 + \mathfrak{M}(\mathsf{A}, \mathsf{M})$ defines a functor from* **LFMon** *to R-***CompTopAlg** *and respectively to* **Grp**.

4. Bialgebras and Hopf algebras

To form the functor $A_R[[-]]\colon \mathbf{Mon}(\mathbb{FinFib}) \to R\text{-}\mathbf{CompTopAlg}$ we let vary the finite decomposition monoid. But one can also see what happens when it remains fixed, and now the commutative R-algebra with unit A varies among the class of all commutative R-algebras with a unit. We show in the Subsection 4.2 that this idea leads to a monoid scheme, i.e., a generalized algebraic monoid, and thus a commutative bialgebra. Using the same kind of approach for the group $1+\mathfrak{M}(\mathsf{A}, \mathsf{M})$ one obtains in the Subsection 4.3 a generalized algebraic group and thus an Hopf algebra. The Subsection 4.1 is devoted to a basic account about monoid and group schemes.

4.1. A short introduction to monoid schemes. The main reference for this subsection is [**7**]. Let R be a commutative ring with a unit. Recall that we denote by $R\text{-}\mathbf{CAlg}_1$ the category of all commutative R-algebras with a unit. Let $F\colon R\text{-}\mathbf{CAlg}_1 \to \mathbf{Set}$ be a functor. When there is a commutative R-algebra $\mathcal{O}(F)$ with a unit such that for each object A of $R\text{-}\mathbf{CAlg}_1$ there exists $\tau_{\mathsf{A}}\colon F(\mathsf{A}) \cong \mathbf{CAlg}_1(\mathcal{O}(F), \mathsf{A})$, the family $(\tau_{\mathsf{A}})_{\mathsf{A}}$ of bijections being a natural transformation, then F is said to be a *representable functor* and the algebra $\mathcal{O}(F)$ is referred to as the[5] *coordinate algebra* of F. Given any other functor $G\colon R\text{-}\mathbf{CAlg}_1 \to \mathbf{Set}$,

[5]It is unique up to an isomorphism of algebras.

the Yoneda lemma (see [**12**]) amounts to say that there exists a bijection from the set $\mathbf{Nat}(F,G)$ of all natural transformations from F to G and the set $G(\mathcal{O}(F))$. Given a natural transformation $\sigma\colon F \Rightarrow G$, then the corresponding element of the set $G(\mathcal{O}(F))$ is given by $\sigma_{\mathcal{O}(F)}(\tau^{-1}_{\mathcal{O}(F)}(\mathsf{id}_{\mathcal{O}(F)}))$ (recall that σ_{A} is a set-theoretic map from $F(\mathsf{A})$ to $G(\mathsf{A})$, and τ_{A} is a bijection from $F(\mathsf{A})$ to $R\text{-}\mathbf{CAlg_1}(\mathcal{O}(F),\mathsf{A})$). When G is also a representable functor, then $\mathbf{Nat}(F,G)$ is in bijection with the set $R\text{-}\mathbf{CAlg_1}(\mathcal{O}(G),\mathcal{O}(F))$ by $(\sigma_{\mathsf{A}})_{\mathsf{A}} \mapsto \mu_{\mathcal{O}(F)}(\sigma_{\mathcal{O}(F)}(\tau^{-1}_{\mathcal{O}(F)}(\mathsf{id}_{\mathcal{O}(F)})))$, where $\mu_{\mathsf{A}}\colon G(\mathsf{A}) \cong R\text{-}\mathbf{CAlg_1}(\mathcal{O}(G),\mathsf{A})$ is the natural bijection associated to G.

Let us consider a functor $\mathsf{M}\colon R\text{-}\mathbf{CAlg_1} \to \mathbf{Mon}$, $\mathsf{A} \mapsto \mathsf{M}(\mathsf{A}) = (M(\mathsf{A}), *_{\mathsf{A}}, e_{\mathsf{A}})$ (respectively, $M\colon R\text{-}\mathbf{CAlg_1} \to \mathbf{Grp}$, $\mathsf{A} \mapsto \mathsf{M}(\mathsf{A}) = (M(\mathsf{A}), *_{\mathsf{A}}, e_{\mathsf{A}}, (-)^{-1}_{\mathsf{A}})$). Composing with the forgetful functor from $\mathbf{Mon}$ (respectively, $\mathbf{Grp}$) to $\mathbf{Set}$ one obtains a functor $M\colon R\text{-}\mathbf{CAlg_1} \to \mathbf{Set}$, $A \mapsto M(A)$, called its *underlying set-valued functor*. M is said to be a *R-monoid* (respectively, *R-group*) whenever the structure maps $(*_{\mathsf{A}})_{\mathsf{A}}\colon M \times M \Rightarrow M$ and $(e_{\mathsf{A}})_{\mathsf{A}}\colon \star \Rightarrow M$ (and furthermore $((-)^{-1}_{\mathsf{A}})_{\mathsf{A}}\colon M \Rightarrow M$ for a R-group) are natural transformations (natural in the algebra A), where one considers the product of functors $M \times M\colon R\text{-}\mathbf{CAlg_1} \to \mathbf{Set}$ given by $(M \times M)(\mathsf{A}) = M(\mathsf{A}) \times M(\mathsf{A})$, and the constant functor $\star\colon R\text{-}\mathbf{CAlg_1} \to \mathbf{Set}$ with value a one-element set $\star$ for any algebra A.

When M is a R-monoid (respectively, R-group) and when its underlying set-valued functor M is a representable functor, then one says that M is a *R-monoid scheme* (respectively, *R-group scheme*).

Remark 4.1. If $\mathcal{O}(M)$ is finitely presented, then M is said to be an *affine algebraic monoid* (respectively, *group*).

Let M be a R-monoid scheme (respectively, a R-group scheme), and let us denote by $\tau_{\mathsf{A}}\colon F(\mathsf{A}) \cong \mathbf{CAlg_1}(\mathcal{O}(M),\mathsf{A})$ the corresponding natural bijection. In this situation one observes that the product functor $M \times M$ is then also a representable functor, represented by $\mathcal{O}(M \times M) = \mathcal{O}(M) \otimes_R \mathcal{O}(M)$, since the usual tensor product $\otimes_R$ over R is the categorical coproduct in the category of commutative R-algebras with a unit, or in other terms, $R\text{-}\mathbf{CAlg_1}(\mathcal{O}(M) \otimes_R \mathcal{O}(M),\mathsf{A}) \cong R\text{-}\mathbf{CAlg_1}(\mathcal{O}(M),\mathsf{A}) \times R\text{-}\mathbf{CAlg_1}(\mathcal{O}(M),\mathsf{A}) \cong M(\mathsf{A}) \times M(\mathsf{A}) = (M \times M)(\mathsf{A})$ for each commutative algebra A with a unit. Also is representable the constant functor $\star$, represented by R, i.e., $\mathcal{O}(\star) = R$, since $R\text{-}\mathbf{CAlg_1}(R,\mathsf{A})$ only consists of a one-element set, namely the scalar multiplication $\alpha \in R \mapsto \alpha 1_{\mathsf{A}}$, because homomorphisms of algebras are assumed to respect the unit element. Hence, by the Yoneda lemma, one has

$$\mathbf{Nat}(M \times M, M) \cong M(\mathcal{O}(M) \otimes_R \mathcal{O}(M)) \cong R\text{-}\mathbf{CAlg_1}(\mathcal{O}(M), \mathcal{O}(M) \otimes_R \mathcal{O}(M))$$

and

$$\mathbf{Nat}(\star, M) \cong M(R) \cong R\text{-}\mathbf{CAlg_1}(\mathcal{O}(R), R)\ .$$

For an R-group scheme, we furthermore have

$$\mathbf{Nat}(M, M) \cong M(\mathcal{O}(M)) \cong R\text{-}\mathbf{CAlg_1}(\mathcal{O}(M), \mathcal{O}(M))\ .$$

In particular, the natural transformation $(*_{\mathsf{A}})_{\mathsf{A}}$ corresponds to

$$\Delta = \tau_{\mathcal{O}(M)\otimes_R\mathcal{O}(M)}(*_{\mathcal{O}(M)\otimes_R\mathcal{O}(M)}(\mu^{-1}_{\mathcal{O}(M)\otimes_R\mathcal{O}(M)}(\mathsf{id}_{\mathcal{O}(M)\otimes_R\mathcal{O}(M)})))\ ,$$

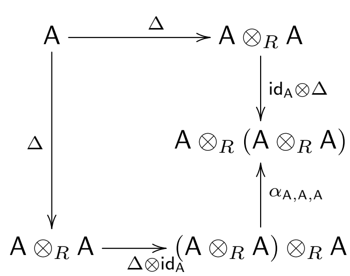
$(e_{\mathsf{A}})_{\mathsf{A}}$ corresponds to $\epsilon = \tau_R(1_R(\pi^{-1}_R(\mathsf{id}_R)))$ (and for an R-group scheme, $((-)^{-1}_{\mathsf{A}})_{\mathsf{A}}$ corresponds to $S = \tau_{\mathcal{O}(M)}((-)^{-1}_{\mathcal{O}(M)}(\tau^{-1}_{\mathcal{O}(M)}(\mathsf{id}_{\mathcal{O}(M)}))))$, where $\mu_{\mathsf{A}}\colon M(\mathsf{A}) \times M(\mathsf{A}) \cong$

R-$\mathbf{CAlg}(\mathcal{O}(M)\otimes_R\mathcal{O}(M),\mathsf{A})$ is given by $\mu_{\mathsf{A}}(x,y)=[\tau_{\mathsf{A}}(x),\tau_{\mathsf{A}}(y)]$ (because $\otimes_R$ is the categorical coproduct in R-$\mathbf{CAlg}_1$, given any homomorphisms of algebras $\phi\colon \mathsf{A}\to\mathsf{C}$ and $\psi\colon\mathsf{B}\to\mathsf{C}$, there is a unique homomorphism of algebras $[\phi,\psi]\colon \mathsf{A}\otimes_R\mathsf{B}\to\mathsf{C}$ such that $[\phi,\psi]\circ q_{\mathsf{A}}=\phi$ and $[\phi,\psi]\circ q_{\mathsf{B}}=\psi$, where $q_{\mathsf{A}}\colon\mathsf{A}\to\mathsf{A}\otimes_R\mathsf{B}$ and $q_{\mathsf{B}}\colon\mathsf{B}\to\mathsf{A}\otimes_R\mathsf{B}$ are the homomorphism of algebras respectively given by $q_{\mathsf{A}}(a)=a\otimes 1_{\mathsf{B}}$ and $q_{\mathsf{B}}(b)=1_{\mathsf{A}}\otimes b$), and $\pi_{\mathsf{A}}\colon\star\cong R$-$\mathbf{CAlg}_1(R,\mathsf{A})$ is given by the unique constant map with value the identity element 1_{A} of the algebra A.

In brief the multiplication $*$, seen as a natural transformation, is associated to a homomorphism of algebras $\Delta\colon\mathcal{O}(M)\to\mathcal{O}(M)\otimes_R\mathcal{O}(M)$, the identity e, also seen as a natural transformation, gives rise to a homomorphism of algebras (also called in this case a *character*) $\epsilon\colon\mathcal{O}(M)\to R$ (and, in case of a R-group scheme, the inverse $(-)^{-1}$ is associated to an endomorphism of algebra $S\colon\mathcal{O}(M)\to\mathcal{O}(M)$). Because $*$, e and $(-)^{-1}$ satisfy some axioms (namely, that of the varieties of all monoids and of all groups), it follows that Δ, ϵ and S also satisfies some laws, namely that of coassociativity, counit and antipode, which are summarized in the following diagrams for any commutative R-algebra A with a unit. So let $\Delta\colon\mathsf{A}\to\mathsf{A}\otimes_R\mathsf{A}$ and $\epsilon\colon\mathsf{A}\to R$ be homomorphisms of algebras. Δ is then referred to as a (coassociative) *coproduct*, ϵ is a *counit*, while a triple (A,Δ,ϵ) is termed as a (commutative) *bialgebra* (see [**17**]), if the two first diagrams below are commutative.

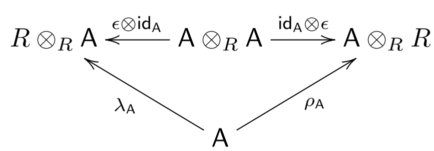
Coassociativity:

(4.1)



Counit:

(4.2)



In the above diagrams, α, λ, ρ are the coherence isomorphisms associated to the monoidal category of all commutative R-algebras with a unit with the usual tensor product $\otimes_R$ of algebras (so $\alpha_{\mathsf{A},\mathsf{B},\mathsf{C}}((a\otimes b)\otimes c)=a\otimes(b\otimes c)$, $\lambda_{\mathsf{A}}(\alpha\otimes a)=\alpha a$, $\rho_{\mathsf{A}}(a\otimes\alpha)=\alpha a$).

Given a bialgebra $(\mathsf{A},\Delta,\epsilon)$, an algebra endomorphism $S\colon\mathsf{A}\to\mathsf{A}$ is said to be an *antipode* when the following diagram commutes in the category of R-modules, where the algebra $\mathsf{A}=(A,m,e)$ is seen as a monoid object in the category R-$\mathbf{Mod}$ with monoidal structure given by $\otimes_R$ (thus in particular, $m\colon A\otimes_R A\to A$, and $e\colon R\to A$ are R-linear maps). In this case, $(\mathsf{A},\Delta,\epsilon,S)$ is said to be a (commutative) *Hopf algebra* (see [**17**]).

Antipode:

(4.3)
$$\begin{array}{ccccc} R & \xleftarrow{\epsilon} & A & \xrightarrow{\epsilon} & R \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle e} \\ A & \xleftarrow[m\circ(S\otimes \mathsf{id}_A)]{} & A\otimes_R A & \xrightarrow[m\circ(\mathsf{id}_A\otimes S)]{} & A \end{array}$$

To summarize, given a R-monoid scheme (respectively, R-group scheme) M, one obtains a structure of bialgebra (respectively, Hopf algebra) on the coordinate algebra of its underlying set-valued functor $\mathcal{O}(M)$. (The proofs may be found in [**7**].)

Remark 4.2. One may note that the diagrams for coassociativity and for counit share many similarities with the diagrams for associativity and two-sided unit (Diagram 2.2). These are actually not just similarities, since it may be proved that a commutative bialgebra is actually a monoid object in the opposite of the monoidal category of commutative algebras with a unit. Moreover, a commutative Hopf algebra is a group object in the same category.

4.2. Bialgebras from finite decomposition monoids. Let us recall that for any algebra $\mathsf{A} = (A, \cdot, 1_{\mathsf{A}})$, $|\mathsf{A}|$ denotes its carrier set (so it is also the carrier set of its underlying R-module A). Let $\mathsf{M} = (M, *, 1)$ be a finite decomposition monoid. Then, the correspondence $(-)_R[[\mathsf{M}]]\colon \mathsf{A} \mapsto (|\mathsf{A}_R[[\mathsf{M}]]|, \star, 1)$ gives rise to a functor from R-$\mathbf{CAlg}_1$ to $\mathbf{Mon}$ (here of course $(|\mathsf{A}_R[[\mathsf{M}]]|, \star, 1)$ denotes the multiplicative underlying monoid of the large algebra $\mathsf{A}_R[[\mathsf{M}]]$) as it is easily checked. Indeed, let $\phi\colon \mathsf{A} \to \mathsf{B}$ be a homomorphism of algebras, and let $f \in \mathsf{A}_R[[\mathsf{M}]]$. Then one defines $\phi(f) \in \mathsf{B}_R[[\mathsf{M}]]$ by $(\hat{\phi}(f))(x) = \phi(f(x))$, $x \in M$. One has $\hat{\phi}(\delta_1)(x) = \phi(\delta_1(x)) = \delta_1(x)$ for each $x \in M$ (since $\phi(1_{\mathsf{A}}) = 1_{\mathsf{B}}$), and $(\hat{\phi}(f \star g))(x) = \phi(\sum_{y*z=x} f(y) \cdot g(z)) = \sum_{y*z=x} \phi(f(y)) \cdot \phi(f(z)) = (\hat{\phi}(f) \star \hat{\phi}(g))(x)$ for each $x \in M$, from which it follows easily that $\phi \mapsto \hat{\phi}$ is the arrow component of a functor that acts on objects as $\mathsf{A} \mapsto \mathsf{A}_R[[\mathsf{M}]]$, and thus one obtains a functor from R-$\mathbf{CAlg}_1$ to $\mathbf{Mon}$ after composition with the forgetful functor from the category R-$\mathbf{Alg}_1$ to $\mathbf{Mon}$. Moreover it defines a R-monoid. Indeed, naturality of the convolution and of the unit follow from the fact that $\hat{\phi}$ is a homomorphism of algebras from $\mathsf{A}_R[[\mathsf{M}]]$ to $\mathsf{B}_R[[\mathsf{M}]]$ for each $\phi \in R$-$\mathbf{CAlg}_1(\mathsf{A}, \mathsf{B})$, and thus preserves both the convolution and the unit.

Lemma 4.3. *The underlying set-valued functor* $|(-)_R[[\mathsf{M}]]|\colon \mathsf{A} \mapsto |\mathsf{A}_R[[\mathsf{M}]]| = |\mathsf{A}|^M$ *is actually a representable functor.*

Proof. Let us consider the free commutative R-algebra $R[M]$ generated by the set M (it is not the monoid algebra $R[\mathsf{M}]$ of the monoid M, but just the polynomial algebra in the members of M as indeterminates). Let us check that $R[M] = \mathcal{O}(|(-)_R[[\mathsf{M}]]|)$. Let A be any commutative R-algebra with a unit. Then, R-$\mathbf{CAlg}_1(R[M], \mathsf{A}) \cong |\mathsf{A}|^M$ (the bijection transforms $\phi \in R$-$\mathbf{CAlg}_1(R[M], \mathsf{A})$ into its restriction $\phi_{|_M}\colon M \to |\mathsf{A}|$). The fact that this bijection is natural follows from well-known properties of left adjoint functors (but may be checked by hand). □

With Lemma 4.3 at hand one can prove the following.

Proposition 4.4. $(-)_R[[\mathsf{M}]]\colon \mathsf{A} \mapsto (|\mathsf{A}_R[[\mathsf{M}]]|, \star, 1)$ *is a R-monoid scheme.*

According to the results of Subsection 4.1, the coordinate algebra $R[M]$ of the underlying set-valued functor of $(-)_R[[\mathsf{M}]]$ is thus a commutative R-bialgebra. Again using Subsection 4.1, one has explicit descriptions for both its coproduct Δ and its counit ϵ, namely

$$\Delta(x) = \sum_{y*z=x} y \otimes z$$

for each $x \in M$, and

$$\epsilon(x) = 0, \ \epsilon(1) = 1$$

for each $x \in M^*$, and both maps are then uniquely extended to homomorphisms of algebras.

4.3. Hopf algebras from locally finite monoids. Let us assume that $\mathsf{M} = (M, *, 1)$ is a locally finite monoid. From Lemma 3.13 it is known that $1 + \mathfrak{M}(\mathsf{A}, \mathsf{M})$ forms a group under convolution for each commutative R-algebra A with a unit. Actually the correspondence $\mathsf{A} \mapsto 1 + \mathfrak{M}(\mathsf{A}, \mathsf{M})$ is functorial. We already know from Subsection 4.2 that $\mathsf{A} \mapsto (|\mathsf{A}_R[[\mathsf{M}]]|, \star, 1)$ is a R-monoid scheme. We also konw that for each $\phi \in R\text{-}\mathbf{CAlg}_1(\mathsf{A}, \mathsf{B})$, $\hat{\phi} \in R\text{-}\mathbf{Alg}_1(\mathsf{A}_R[[\mathsf{M}]], \mathsf{B}[[\mathsf{M}]])$ and thus gives rise to a morphism of group $\mathsf{U}(\hat{\phi})\colon \mathsf{U}(\mathsf{A}_R[[\mathsf{M}]]) \to \mathsf{U}(\mathsf{B}_R[[\mathsf{M}]])$ just by restriction. Let $f \in \mathfrak{M}(\mathsf{A}, \mathsf{M})$. Then, $\mathsf{U}(\hat{\phi})(1 + f) = \hat{\phi}(1) + \hat{\phi}(f) = 1 + \hat{\phi}(f)$. Since $f(1) = 0$, one has $(\hat{\phi}(f))(1) = \phi(f(1)) = \phi(0) = 0$, so that $\hat{\phi}(f) \in 1 + \mathfrak{M}(\mathsf{B}, \mathsf{M})$. This actually defines an R-group. Naturality of the multiplication and of the unit follow from the fact that $(-)_R[[\mathsf{M}]]\colon A \mapsto (|A_R[[\mathsf{M}]]|, \star, 1)$ is a monoid scheme. It suffices to check naturality of the inversion map but it follows from the fact that $\mathsf{U}(\hat{\phi})$ is a homomorphism of groups for each $\phi \in R\text{-}\mathbf{CAlg}_1(\mathsf{A}, \mathsf{B})$.

Let us consider the polynomial algebra $R[M^*]$, where $M^* = M \setminus \{1\}$. One has $R\text{-}\mathbf{CAlg}(R[M^*], \mathsf{A}) \cong |\mathsf{A}|^{M^*}$ but any member of the group $1 + \mathfrak{M}(\mathsf{A}, \mathsf{M})$ is a map from M to $|\mathsf{A}|$ that takes the value 1_A at the identity 1 of M, thus it is completely determined by its values on M^*, proving that the underlying set of $1 + \mathfrak{M}(\mathsf{A}, \mathsf{M})$ is equipotent to $|\mathsf{A}|^{M^*}$ by a bijection $\tau_\mathsf{A}(1 + f) = f$, $f \in \mathfrak{M}(\mathsf{A}, \mathsf{M})$. The naturality of this bijection is obvious because $\hat{\phi}(1 + f) = 1 + \hat{\phi}(f)$ for each $\phi \in R\text{-}\mathbf{CAlg}_1(\mathsf{A}, \mathsf{B})$ and each $f \in \mathsf{A}_R[[\mathsf{M}]]$. This shows at once that the underlying set-valued functor is representable with coordinate algebra $R[M^*]$, so the functorial correspondence $\mathsf{A} \mapsto 1 + \mathfrak{M}(\mathsf{A}, \mathsf{M})$ is a R-group scheme. It follows from Subsection 4.1 that $R[M^\star]$ admits a structure of a Hopf algebra induced by the Yoneda lemma from the structure of the R-group scheme. Again using Subsection 4.1, one has explicit descriptions for its coproduct Δ, its counit ϵ, namely

$$\Delta(x) = \sum_{\substack{y \neq 1, z \neq 1 \\ y*z=x}} y \otimes z$$

for each $x \in M^*$ (recall from Section 3.3 that M^* forms a semigroup), and

$$\epsilon(x) = 0$$

for each $x \in M^*$. For the antipode S, we let $\zeta\colon x \in M \mapsto 1_R$ be the usual zêta function of the monoid M, it belongs to $1 + \mathfrak{M}(R[M], \mathsf{M})$ since $\zeta = 1 + \delta_{M^*}$, where $\delta_{M^*}(1) = 0$ and $\delta_{M^*}(x) = 1_R$ for each $x \in M^*$, and its inverse is the Möbius function $\mu = (-\delta_{M^*})^\star$ (see [**5**]). Then, $S(x) = \mu(x)$ for each $x \in M^*$. The maps Δ, ϵ, S are then uniquely extended to homomorphisms of algebras.

REMARK 4.5. The construction of the Hopf algebra associated to the group scheme $1 + \mathfrak{M}(\mathsf{A}, \mathsf{M})$ is known in the case $\mathsf{M} = \mathbb{N}$ (see [**4**]), although it is not stated that $1 + \mathfrak{M}(\mathsf{A}, \mathsf{M})$ indeed is a group scheme. Nevertheless this construction is only possible when the base ring R is actually a field (since it uses a dualization process), while we avoid this technical restriction by using the Yoneda lemma. The same limitation appears in [**10**].

EXAMPLE 4.6. Let $L(X; C)$ be the free partially commutative monoid generated by the commutation alphabet (X, C). It is a locally finite monoid. Let us assume that $R = \mathbb{Z}$. Then, using the computation of the Möbius function of $L(X; C)$ in [**5**] one has for each $x \in L(X; C)^*$, $S(x) = -1$. More generally if M is any quotient monoid of a free monoid by a multi-homogeneous congruence, then its Möbius function satisfies $\mu(x) = -1$, where x is the image of a generator. In the case of the monoid $\mathbb{N}$, one recovers for $\mathbb{C}[\mathbb{N}^*]$ the (complex) Hopf algebra of representative functions on the group of invertible formal power series in one variable (see [**4**]).

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UNIVERSITÉ PARIS 13, SORBONNE PARIS CITÉ, LIPN, CNRS (UMR 7030), VILLETANEUSE, FRANCE
E-mail address: laurent.poinsot@lipn.univ-paris13.fr