Differential (Monoid) Algebra and More

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Abstract. On any set X may be defined the free algebra $R\langle X\rangle$ (respectively, free commutative algebra $R[X]$) with coefficients in a ring R. It may also be equivalently described as the algebra of the free monoid X^* (respectively, free commutative monoid $\mathcal{M}(X)$). Furthermore, the algebra of differential polynomials $R\{X\}$ with variables in X may be constructed. The main objective of this contribution is to provide a functorial description of this kind of objects with their relations (including abelianization and unitarization) in the category of differential algebras, and also to introduce new structures such as the differential algebra of a semigroup, of a monoid, or the universal differential envelope of an algebra.

Keywords: Differential algebra, monoid algebra, free algebra, category theory.

1 Introduction

On any set X may be defined the free algebra $R\langle X\rangle$ of non-commutative polynomials with variables in X and coefficients in a (commutative and unital) ring R. This is the tensor algebra of the free module RX with basis X. Its abelianization, the symmetric algebra of RX, corresponds to the free commutative algebra $R[X]$ of polynomials with variables in X. Moreover each of these algebras may be equivalently defined as the algebra of a monoid. Thus $R\langle X\rangle$ is the algebra of the monoid X^* while $R[X]$ is the algebra of the free commutative monoid $\mathcal{M}(X)$ over X. Replacing X^* by the free semigroup X^+ over X (thus X^+ is the set of non-empty words over X and X^* is obtained from X^+ by adding an identity element), we may speak about the algebra $R\langle X\rangle^+$ of the semigroup X^+, which is the free non-unital algebra over X. Its abelianization, denoted by $R[X]^+$, is then the free commutative non-unital algebra over X and is also characterized as the semigroup algebra of the abelianization $\mathcal{S}(X)$ of X^+, namely the free commutative semigroup over X. The algebras $R\langle X\rangle$ and $R[X]$ are recovered from $R\langle X\rangle^+$ and $R[X]^+$ respectively by a free adjunction of an identity that turns any non-unital algebra into a unital one in the usual way. Thus, $R\langle X\rangle \cong R\langle X\rangle^+ \oplus R$ and $R[X] \cong R[X]^+ \oplus R$ as modules over R, where $R\langle X\rangle^+$ and $R[X]^+$ are the kernels of the augmentation maps that send a polynomial (commutative or not) to its constant term (the coefficient of the empty word in the polynomial under

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consideration). Now, again for any set X we may define the algebra $R\{ X \}$ of differential (commutative) polynomials over R in the standard way. It consists of the polynomial algebra in the variables $x^{(i)}$, $x \in X$, $i \geq 0$, together with a derivation ∂ that acts on the basis elements $x^{(i)}$ by $\partial(x^{(i)}) = x^{(i+1)}$ for every $x \in X$, and $i \geq 0$. (Here R is just a ring[1], but the case of differential polynomials over a differential ring is almost given by the same construction, and is treated in this contribution.) By analogy to the above algebraic case we may ask a few questions. (1) Is there a non-commutative differential algebra, say $R\{\!\{ X \}\!\}$, for which $R\{ X \}$ is the abelianization? (2) What kind of universal properties do these algebras satisfy? (3) Does the algebra $R\{ X \}$ (and $R\{\!\{ X \}\!\}$ if it exists) may be interpreted, in the differential setting, as the algebra of some monoid? (4) What is the nature of the relations between the algebras $R\{\!\{ X \}\!\}$ and $R\langle X \rangle$, and the algebras $R\{ X \}$ and $R[X]$?

The main objective of this contribution is to give precise answers to these questions. In order to make this program possible we adopt a functorial point of view. More precisely the relations between algebraic and differential structures are described through the general ideas of forgetful functors and left adjoints. This allows us to characterize the objects by universal properties. This contribution provides a large panorama of structures and connections between them in the differential setting that generalizes some usual algebraic objects such as the algebra of a monoid, the abelianization functor between algebras, the free adjunction of a unit, and differential polynomials. In this way the usual differential polynomials will be seen as a particular instance of what we call the differential algebra of a monoid, but also as the differential algebra freely generated by the usual (non-differential) commutative polynomials. Moreover non-commutative versions are also provided together with their relations with their commutative counterparts.

A reader interested in differential algebra might not guess why these new objects and constructions could be useful. However we think that it is rather natural to consider generalized differential equations in variables that satisfy some non trivial relations (not just commutativity), exactly as our differential algebra of a semigroup for which for instance the theory of differential Gröbner bases [15], *i.e.*, Gröbner bases for differential ideals, or G.M. Bergman's notion of polynomial reduction systems [1], should be extended. As the functorial viewpoint on classical algebraic geometry was so successful (*e.g.*, [7]), we hope that such an approach to differential algebraic geometry may lead to new progress in this area [12]. We also mention a recent interest about non-commutative differential equations [8], and also the author's paper [16] where is defined and studied a universal differential enveloping algebra for a Lie algebra. In both cases a functorial approach should be relevant to obtain new results.

[1] It is actually a differential ring with the zero derivation, so that it may be referred to as a "trivial" differential ring.

State of the Art and Overview of the New Results

What is well-known for a long time in the area of differential algebra related to the present work is the construction of the ring of differential commutative polynomials (over a differential ring). Not as well-known is the construction of its non-commutative counterpart [5] which is quite more recent. And that's all, up to our knowledge. On the other side much more algebraic (and not differential) constructions, again related to our work, are classical: the ring of a semigroup or a monoid, the unitarization of semigroups or non-unital rings, the abelianization of these various structures. In brief, the paper presents a variety of new constructions in the spirit of differential polynomials for obtaining differential rings from semigroups and monoids. These constructions may be seen as the counterparts of the algebraic ones in the category of differential algebras. Sections 2 and 3 are devoted to basic definitions related to universal problems and to differential algebra respectively. The other sections contain the new constructions and what connects them. In Section 4 is introduced the concept of the non-commutative differential algebra $R\wr S \wr$ of a semigroup or a monoid S (Lemma 4) that generalizes the usual construction of the algebra $R[S]$ of S, over some commutative base ring R, as the solution of some universal problem (or equivalently as a free construction). Again in the same section the commutative counterpart $R\{\, S\,\}$ is defined for a commutative S (Lemma 7). When S is the free commutative monoid over X, then $R\{\, S\,\}$ is the usual differential ring of commutative differential polynomials with variables in X (as already mentioned above), and for S the free monoid over X, $R\wr S \wr$ is the ring of non-commutative differential polynomials with variables in X. It is also proved that the unitarization (as a differential algebra) $R\wr S \wr_1$ of $R\wr S \wr$ is (isomorphic to) the differential algebra $R\wr S^1 \wr$ of the unitarization of the semigroup S (Lemma 6, and Lemma 9 for the commutative case), and the abelianization (as a differential algebra) $\mathscr{Ab}(R\wr S \wr)$ of $R\wr S \wr$ is (isomorphic to) $R\{\, \mathscr{Ab}(S)\,\}$ (Lemma 11), where $\mathscr{Ab}(S)$ is the usual abelianization of a semigroup or a monoid. In Section 5 is proved the existence of a universal differential envelope $\mathcal{D}(A)$ of a usual algebra A, *i.e.*, the free differential algebra generated by a usual algebra both in the commutative and non-commutative cases, and unital and non-unital cases (Lemmas 15 and 16). It is proved that the differential algebra of a semigroup or monoid is the universal differential envelope of its usual algebra (Corollaries 1 and 2). Also are explained the relations with abelianization (Lemma 17) and unitarization (Lemma 18). Finally in the Section 6 are extended some of these results to the setting of differential (unital) algebras over a differential base ring (rather than over a usual base ring) that extends the notion of differential commutative polynomials over a differential ring. The paper concludes with an overview of the constructions: when fitted all together the results from this paper lead to a large commutative diagram of functors (13) between the categories of unital/non-unital, commutative/non-commutative, differential/non-differential algebras (different construction paths lead to isomorphic objects).

2 Some Introducing Remarks about Universal Problems

Most of the constructions we deal with in this paper are actually special instances of a general category-theoretic fact that algebraic functors between varieties of (universal) algebras admit a left adjoint (see for instance Corollary 8.17 in [2]). Nevertheless this result is too general to provide any insight about the explicit constructions. Thus, we follow the more classical point of view of universal problems.

Universal problems and properties are so important to this contribution that they deserve their own section. The basic definitions and results, some of which being briefly recalled below, about category theory (such as functors, full, faithful and forgetful functors, left adjoints) may be found in [14] and other (such as universal problems and universal properties) may be found in a full generality in [17]. Let $\mathcal{C}$ be a category. We denote by $\mathcal{C}(A,B)$ the class of morphisms from an object A to an object B of $\mathcal{C}$, and the identity morphism of A is denoted by id_A. Let $\mathcal{D}$ be also a category, and let $U\colon \mathcal{C} \to \mathcal{D}$ be a functor. Let X be an object of $\mathcal{D}$. An object S_X of $\mathcal{C}$ together with a morphism $\eta_X : X \to U(S_X)$ is said to be a *solution* to the *universal problem* associated to U (or *satisfies the universal problem*) if for every object A of $\mathcal{C}$ and every morphism (in $\mathcal{D}$) $\phi\colon X \to U(A)$, there is a unique morphism $\widehat{\phi}\colon S_X \to A$ such that $U(\widehat{\phi}) \circ \eta_X = \phi$. We also say that S_X (with a given η_X) has a *universal property* (associated to U), and the associated map η_X is referred to as the *canonical arrow* (or *map*) or the *natural arrow* (or *map*) associated to X and the solution S_X. We mention that the unique arrow $\widehat{\phi}\colon S_X \to A$, corresponding to a morphism $\phi\colon X \to U(A)$, is also referred to as a *canonical map* (or *arrow* or *morphism*).

Let us assume that for a same object X one has two solutions say (S_X, η_X) and (T_X, ν_X). Then, S_X and T_X are isomorphic as objects of $\mathcal{C}$. Indeed, because of the universal property of S_X, there is a unique morphism $\widehat{\nu}_X : S_X \to T_X$ (in $\mathcal{C}$) such that $U(\widehat{\nu}_X) \circ \eta_X = \nu_X$. Similarly, because of the universal property of T_X, there is a unique morphism $\widetilde{\eta}_X : T_X \to S_X$ such that $U(\widetilde{\eta}_X) \circ \nu_X = \eta_X$. Thus, $U(\widetilde{\eta}_X \circ \widehat{\nu}_X) \circ \eta_X = U(\widetilde{\eta}_X) \circ U(\widehat{\nu}_X) \circ \eta_X = U(\widetilde{\eta}_X) \circ \nu_X = \eta_X$, and conversely, $U(\widehat{\nu}_X \circ \widetilde{\eta}_X) \circ \nu_X = U(\widehat{\nu}_X) \circ U(\widetilde{\eta}_X) \circ \nu_X = U(\widehat{\nu}_X) \circ \eta_X = \nu_X$. But we also have $U(id_{S_X}) \circ \eta_X = id_{U(S_X)} \circ \eta_X = \eta_X$, and $U(id_{T_X}) \circ \nu_X = id_{U(T_X)} \circ \nu_X = \nu_X$. By the universal property of S_X this implies that $id_{S_X} = \widetilde{\eta}_X \circ \widehat{\nu}_X$ (because there is a unique morphism $\psi\colon S_X \to S_X$ such that $U(\psi) \circ \eta_X = \eta_X$), and the universal property of T_X implies that $U(id_{T_X}) = \widehat{\nu}_X \circ \widetilde{\eta}_X$ (because there is a unique morphism $\theta\colon T_X \to T_X$ such that $U(\theta) \circ \nu_X = \nu_X$). Thus we conclude that $S_X \cong T_X$ (in $\mathcal{C}$). As a consequence, to prove that an object of $\mathcal{C}$ is isomorphic to another one that we know to satisfy some universal problem, it is sufficient to prove that it satisfies the same universal problem.

Remark 1. The kind of proof as above (using uniqueness of some morphism) is very usual in category theory. In this contribution we also take it as usual and hereafter we don't develop all the details of such a proof (see for instance the end of the proof of Lemma 11, which is based on exactly the same kind of reasoning, where we only mention "usual categorical arguments" referring by this sentence to the above kind of proof).

A large source of such universal problems comes from (categorical) adjunctions. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. Let $U\colon \mathcal{C} \to \mathcal{D}$ and $F\colon \mathcal{D} \to \mathcal{C}$ be two functors. We say that F is a *left adjoint* to U (or that (U, F) is an *adjunction*) if there exists a natural bijection $\psi_{X,A}\colon \mathcal{D}(X, UA) \cong \mathcal{C}(FX, A)$ (where A is an object of $\mathcal{C}$ and X and object of $\mathcal{D}$). In particular, let $A = FX$ for some object X of $\mathcal{D}$, then $\mathcal{D}(X, UFX) \cong \mathcal{C}(FX, FX)$, and we denote by $\eta_X\colon X \to UFX$ the arrow $\psi^{-1}_{X,FX}(id_{FX})$.

Example 1. Let $\mathcal{C}$ be the category of groups, and $\mathcal{D}$ be that of sets. Let U be the obvious forgetful functor. Then, F is the free group on a set construction, and for every set X, η_X is the map that identifies a member of X with the corresponding free generator of FX.

In such a situation, FX, with η_X, is a solution to the universal problem associated to U: let $\phi\colon X \to U(A)$ be a morphism in $\mathcal{D}$ (where A is an object of $\mathcal{D}$), then the unique morphism (in $\mathcal{C}$) $\widehat{\phi}$ from FX to A is given by $\psi_{X,A}(\phi)$. A full detailed account on universal properties is given in [17].

3 Basic Definitions from Differential Algebra

In this contribution, every ring is assumed to be both unital and commutative. Nevertheless, given such a commutative ring R with a unit, commutativity is not assumed for R-algebras, nor it is assumed that they possess an identity (but obviously they are assumed to be associative). We refer to [13,18,19] concerning notions about the field of differential algebra which are recalled hereafter. The categories of R-algebras, R-algebras with a unit, commutative R-algebras and commutative R-algebras with a unit are denoted respectively by $R\text{-}\mathcal{Alg}$, $R\text{-}\mathcal{Alg}_1$, $R\text{-}\mathcal{CAlg}$ and $R\text{-}\mathcal{CAlg}_1$. Homomorphisms between algebras (commutative or not) with a unit are assumed to respect the units.

Let R be a ring, and A be a R-algebra. A *R-derivation* (or *R-linear derivation*) of A is a R-linear endomorphism $\partial\colon A \to A$ that satisfies Leibniz rule, *i.e.*, for every $a, b \in A$, $\partial(ab) = \partial(a)b + a\partial(b)$. In particular, if A is an algebra with a unit 1_A, then $\partial(1_A) = 0$. An algebra A together with a R-linear derivation ∂ is called a *differential R-algebra*. It is said to be commutative (respectively, unital) when the underlying algebra A is so.

Given two differential algebras (over R) (A, ∂_A) and (B, ∂_B), a homomorphism of differential algebras $\phi\colon (A, \partial_A) \to (B, \partial_B)$ is a homomorphism of algebras from A to B that commutes with the derivations $\phi \circ \partial_A = \partial_B \circ \phi$. (In particular, if A and B are both unital, then it is assumed that $\phi(1_A) = 1_B$.) The categories of differential R-algebras, differential R-algebras with a unit, commutative differential R-algebras and commutative differential R-algebras with a unit are denoted respectively by $R\text{-}\mathcal{DiffAlg}$, $R\text{-}\mathcal{DiffAlg}_1$, $R\text{-}\mathcal{CDiffAlg}$ and $R\text{-}\mathcal{CDiffAlg}_1$. It is clear that $R\text{-}\mathcal{CDiffAlg}$ (respectively, $R\text{-}\mathcal{CDiffAlg}_1$) is a full subcategory of $R\text{-}\mathcal{DiffAlg}$ (respectively, $R\text{-}\mathcal{DiffAlg}_1$), while $R\text{-}\mathcal{DiffAlg}_1$ (respectively, $R\text{-}\mathcal{CDiffAlg}_1$) is a (non-full) subcategory of R-$\mathcal{DiffAlg}$ (respectively, R-$\mathcal{CDiffAlg}$). We observe that any usual R-algebra (commutative or not, unital or not) is also a differential R-algebra when equipped with the zero derivation. Actually these define full embeddings (see Lemma 13 in Subsection 4.6) given by $J\colon A \mapsto (A, 0)$ for every algebra A. Conversely, there are obvious forgetful functors (that "forgets" the derivation) from R-$\mathcal{DiffAlg}$ to R-$\mathcal{Alg}$ (respectively, from R-$\mathcal{DiffAlg}_1$ to R-$\mathcal{Alg}_1$, from R-$\mathcal{CDiffAlg}$ to R-$\mathcal{CAlg}$, and from R-$\mathcal{CDiffAlg}_1$ to R-$\mathcal{CAlg}_1$).

Example 2. Let X be any set, and let us denote the element $(x, i) \in X \times \mathbb{N}$ by $x^{(i)}$. The algebra $R\{X\}$ of differential polynomials over X (see [13]) is the usual algebra of polynomials $R[X \times \mathbb{N}]$ generated by all $x^{(i)}$'s together with the derivation ∂ such that $\partial(x^{(i)}) = x^{(i+1)}$ for each $x \in X$ and each $i \geq 0$. More generally, if (A, d_A) is a commutative and unital differential R-algebra, then we define similarly $A\{X\}$ with the derivation such that $\partial(x^{(i)}) = x^{(i+1)}$ for each $x \in X$ and each $i \geq 0$ that extends that of A, *i.e.*, $\partial(\alpha x^{(i)}) = d_A(\alpha)x^{(i)} + \alpha x^{(i+1)}$ for every $\alpha \in A$. This means that $(A\{X\}, \partial)$ is a differential R-algebra (and we recover $R\{X\}$ by considering for (A, d_A) the differential algebra $(R, 0)$), but not a differential A-algebra (see also Section 6).

Given a differential R-algebra (A, ∂), a (two-sided) ideal I of A is said to be a *differential ideal* when $\partial(I) \subseteq I$. Then, the quotient algebra $A/_I$ becomes a differential algebra in a natural way: there is a unique derivation $\widetilde{\partial}$ on $A/_I$ such that $\widetilde{\partial} \circ \pi_I = \pi_I \circ \partial$, where $\pi_I\colon A \to A/_I$ is the natural epimorphism (it becomes a homomorphism of differential algebras when $A/_I$ is equipped with $\widetilde{\partial}$).

4 Differential Semigroup Algebra

The well-known algebra $R\{X\}$ of differential polynomials (see for instance [13] and example 2) on a set X is the basis of the developments presented in this contribution. One of our objectives will be to prove that it is actually the differential algebra of some monoid, and also to present its non-commutative counterpart. This section is devoted to the construction of the universal differential associative envelope of a semigroup or a monoid, *i.e.*, the differential algebra freely generated by a semigroup or a monoid, which extends the usual structure of algebra of a semigroup. Informally, and this is formally proved in Section 5, the former is obtained by adding all derivatives of members of the later. The cases of semigroups, commutative or not, and monoids, commutative or not, are presented hereafter together with their main connections (abelianization functors, and adjunction of a unit).

4.1 Preliminaries on Free Partially Commutative Structures

In this subsection is adopted the language of free partially commutative structures (see for instance [4,9,10,20]) that allows a kind of interpolation between non-commutativity and full commutativity, and which is recalled hereafter. A *commutation alphabet* (X, θ) is a set X with an irreflexive relation $\theta \subseteq X^2$. Let S be either a semigroup, a monoid, an algebra or an algebra with a unit (over a ring R). Let $\phi \colon X \to S$. It is said to *respect the commutations* if for every $(x, y) \in \theta$, $\phi(x)\phi(y) = \phi(y)\phi(x)$. There exists a *free partially commutative* semigroup (respectively, monoid, algebra, algebra with a unit) over (X, θ) denoted by $\mathcal{S}(X, \theta)$ (respectively, $\mathcal{M}(X, \theta)$, $R\langle X, \theta\rangle^+$ and $R\langle X, \theta\rangle$) characterized as the solution of the following universal problem: there is a one-to-one map $i_{(X,\theta)}$ from X to the free partially commutative structure $\mathcal{S}(X, \theta)$ (respectively, $\mathcal{M}(X, \theta)$, $R\langle X, \theta\rangle^+$ and $R\langle X, \theta\rangle$), called the *canonical map* (see Section 2), that respects the commutations, such that for every semigroup (respectively, monoid, algebra, algebra with a unit) S and every map $\phi \colon X \to S$ that respects the commutations, then there is a unique homomorphism $\widehat{\phi}$ from $\mathcal{S}(X, \theta)$ (respectively, $\mathcal{M}(X, \theta)$, $R\langle X, \theta\rangle^+$ and $R\langle X, \theta\rangle$) to S such that $\widehat{\phi}(i_{(X,\theta)}(x)) = \phi(x)$ for every $x \in X$.

If one denotes by X^+ (respectively, X^*, $R\langle X\rangle^+$ and $R\langle X\rangle$) the free semigroup (respectively, monoid, algebra, algebra with a unit) over X, then $\mathcal{S}(X, \theta)$ (respectively, $\mathcal{M}(X, \theta)$) is obtained as the quotient semigroup (monoid) by the least congruence on X^+ (respectively, on X^*) generated by the (xy, yx)'s, $(x, y) \in \theta$, while $R\langle X, \theta\rangle^+$ (respectively, $R\langle X, \theta\rangle$) is obtained as the quotient algebra of $R\langle X\rangle^+$ (respectively, $R\langle X\rangle$) by the two-sided ideal generated by $\{\, xy - yx \colon (x, y) \in \theta \,\}$. The map $i_{(X,\theta)}$ is given as the composition between the natural epimorphism associated to the quotient and the usual embedding of the alphabet X into the free structure (it is indeed proved to be one-to-one, see for instance [9]). In what follows the alphabet X is identified with its image by $i_{(X,\theta)}$ into $\mathcal{S}(X, \theta)$, respectively $\mathcal{M}(X, \theta)$, $R\langle X, \theta\rangle^+$, $R\langle X, \theta\rangle$.

For any semigroup S we define the monoid $S^1 = S \sqcup \{\, 1 \,\}$ obtained by free adjunction of a unit (where $\sqcup$ denotes the disjoint sum) with the obvious extension of the multiplication in S. It satisfies the following universal problem: for any monoid M and any homomorphism of semigroups $\phi \colon S \to M$ there is a unique homomorphism of monoids $\phi^1 \colon S^1 \to M$ such that $\phi^1(x) = \phi(x)$ for every $x \in S$ (and of course $\phi^1(1) = 1$). Similarly, let A be a non-unital R-algebra. Let $A_1 = A \oplus R$ (direct sum of R-modules). We define a multiplication on A_1 in a usual way by $(x, \alpha)(y, \beta) = (xy + \alpha y + \beta x, \alpha\beta)$. Then, A_1 becomes a R-algebra with a unit and A is a two-sided ideal, kernel of the augmentation map $A_1 \to R$ that maps (x, α) onto α. The algebra A_1 also is a solution of a universal problem: let B be any R-algebra with a unit, and $\phi \colon A \to B$ be a homomorphism of algebras. Then, there is a unique homomorphism of algebras with unit $\phi_1 \colon A_1 \to B$ such that $\phi_1(x, 0) = \phi(x)$ for every $x \in A$ (and therefore, $\phi_1(x, \alpha) = \phi(x) + \alpha 1_B$).

It can be proved that $\mathcal{M}(X, \theta)$ may be obtained from $\mathcal{S}(X, \theta)$ by a free adjunction of an identity ϵ (the empty word[2]), that is $\mathcal{M}(X, \theta) = \mathcal{S}(X, \theta)^1 = \mathcal{S}(X, \theta) \sqcup$

[2] More precisely, the image of the empty word by the quotient map from X^* to $\mathcal{M}(X, \theta)$.

$\{\epsilon\}$. Similarly, $R\langle X,\theta\rangle = R\langle X,\theta\rangle_1^+ = R\langle X,\theta\rangle^+ \oplus R$. Besides[3], $R\langle X,\theta\rangle^+ = R[\mathcal{S}(X,\theta)]$ and $R\langle X,\theta\rangle = R[\mathcal{M}(X,\theta)]$, where $R[S]$ denotes the usual algebra of the semigroup (respectively, monoid) S.

If $\theta = \emptyset$, then $\mathcal{S}(X,\emptyset)$ (respectively, $\mathcal{M}(X,\emptyset), R\langle X,\emptyset\rangle^+, R\langle X,\emptyset\rangle$) is the free non-commutative semigroup (respectively, monoid, algebra, algebra with a unit) X^+ (respectively, $X^*, R\langle X\rangle^+, R\langle X\rangle$) on the alphabet X. Let $\Delta_X = \{(x,y) \in X^2 \colon x \neq y\}$. Then, $\mathcal{S}(X,\Delta_X)$ (respectively, $\mathcal{M}(X,\Delta_X), R\langle X,\Delta_X\rangle^+, R\langle X,\Delta_X\rangle$) is the free commutative semigroup (respectively, monoid, algebra, algebra with a unit) $\mathcal{S}(X)$ (respectively, $\mathcal{M}(X), R[X]^+, R[X]$) on the alphabet X.

4.2 Extension of Maps as Derivations

In this subsection are given three lemmas that allow us to construct derivations from maps defined on a commutation alphabet which respect the commutations (see also [3], Lemme 4, for a similar result).

Lemma 1. *Let R be a ring, and X be any set. Let $\partial_X \colon X \to R\langle X\rangle^+$ be a set-theoretic map. Then there exists a unique map $\partial \colon X^+ \to R\langle X\rangle^+$ such that $\partial(x) = \partial_X(x)$ for every $x \in X$, and $\partial(uv) = \partial(u)v + u\partial(v)$ for every $u,v \in X^+$. Moreover, ∂ extends uniquely to a map $\partial \colon X^* \to R\langle X\rangle$ such that for every $u,v \in X^*$, $\partial(uv) = \partial(u)v + u\partial(v)$. This map $\partial \colon X^* \to R\langle X\rangle$ may also be equivalently defined as the unique extension of ∂_X such that for every $u,v \in X^*$, $\partial(uv) = \partial(u)v + u\partial(v)$.*

Proof. We first define $\partial \colon X^+ \to R\langle X\rangle^+$ by induction on the length of a word. We have $\partial(x) = \partial_X(x)$ and $\partial(xu) = \partial_X(x)u + x\partial(u)$ for every $x \in X$, $u \in X^+$. Let us prove that for every words $u,v \in X^+$, $\partial(uv) = \partial(u)v + u\partial(v)$ by induction on the length of uv. We have $\partial(xv) = \partial_X(x)v + x\partial(v) = \partial(x)v + x\partial(v)$ for every $x \in X$, $v \in X^+$. We have

[3] Let us sketch the proof of the fact that $R\langle X,\theta\rangle^+ = R[\mathcal{S}(X,\theta)]$. We check that $R\langle X,\theta\rangle^+$ satisfies the same universal property of the algebra $R[\mathcal{S}(X,\theta)]$ of the semigroup $\mathcal{S}(X,\theta)$ which implies that they are isomorphic. Let A be a R-algebra, and let $\phi \colon \mathcal{S}(X,\theta) \to A$ be a homomorphism of semigroups (A is seen as a semigroup with respect to its multiplicative structure). Then, ϕ and $\phi_0 = \phi \circ i_{(X,\theta)}$, with $i_{(X,\theta)} \colon X \to \mathcal{S}(X,\theta)$ being the canonical map, both respect the commutations. Thus there is a unique homomorphism of algebras $\widehat{\phi_0} \colon R\langle X,\theta\rangle^+ \to A$ such that $\widehat{\phi_0} = \phi_0$. Let $j_{\mathcal{S}(X,\theta)} \colon \mathcal{S}(X,\theta) \to R\langle X,\theta\rangle^+$ be the obvious map that identifies a member w of the semigroup with the polynomial w. Of course, it is a homomorphism of semigroups, and $\widehat{\phi_0} \circ j_{\mathcal{S}(X,\theta)} = \phi$ (indeed, let $i'_{(X,\theta)} \colon X \to R\langle X,\theta\rangle^+$ be the canonical map, then it easy to see that $j_{\mathcal{S}(X,\theta)} \circ i_{(X,\theta)} = i'_{(X,\theta)}$, and then $\widehat{\phi_0} \circ j_{\mathcal{S}(X,\theta)} \circ i_{(X,\theta)} = \widehat{\phi_0} \circ i'_{(X,\theta)} = \phi_0 = \phi \circ i_{(X,\theta)}$ so that $\widehat{\phi_0} \circ j_{\mathcal{S}(X,\theta)} = \phi$ by uniqueness of such a map). Let $\psi \colon R\langle X,\theta\rangle^+ \to A$ be a homomorphism of algebras such that $\psi \circ j_{\mathcal{S}(X,\theta)} = \phi$. Then again $\psi \circ j_{\mathcal{S}(X,\theta)} \circ i_{(X,\theta)} = \psi \circ i_{(X,\theta)} = \phi_0$, so by the universal property of $R\langle X,\theta\rangle^+$, $\psi = \widehat{\phi_0}$. Thus, we can take $R\langle X,\theta\rangle^+$ and $R[\mathcal{S}(X,\theta)]$ as equal.

$$\begin{aligned} \partial(xuv) &= \partial(x)uv + x\partial(uv) \\ &= \partial(x)uv + x(\partial(u)v + u\partial(v)) \\ &\quad \text{(by induction)} \\ &= (\partial(x)u + x\partial(u))v + xu\partial(v) \\ &= \partial(xu)v + xu\partial(v) \end{aligned} \tag{1}$$

for every $x \in X$, $u, v \in X^+$. Now, we extend ∂ as a map $\partial\colon X^* \to R\langle X\rangle$ by setting $\partial(\epsilon) = 0$. This extension is easily seen to be unique and that for every $u, v \in X^*$, $\partial(uv) = \partial(u)v + u\partial(v)$. It is also quite clear that it is the unique unique extension of ∂_X such that for every $u, v \in X^*$, $\partial(uv) = \partial(u)v + u\partial(v)$. □

Lemma 2. *Let R be a ring, (X, θ) be any commutation alphabet, and $\partial_X : X \to R\langle X, \theta\rangle^+$ be a set-theoretic map such that for every $(x, y) \in \theta$, $\partial_X(x)y = y\partial_X(x)$. Then, there exists a unique map $\partial\colon \mathcal{S}(X, \theta) \to R\langle X, \theta\rangle^+$ such that $\partial(x) = \partial_X(x)$ for every $x \in X$, and $\partial(uv) = \partial(u)v + u\partial(v)$ for every $u, v \in \mathcal{S}(X, \theta)$. Moreover, ∂ extends uniquely to a map $\partial\colon \mathcal{M}(X, \theta) \to R\langle X, \theta\rangle$ such that for every $u, v \in \mathcal{M}(X, \theta)$, $\partial(uv) = \partial(u)v + u\partial(v)$. This map $\partial\colon \mathcal{M}(X, \theta) \to R\langle X, \theta\rangle$ may also be equivalently defined as the unique extension of ∂_X such that for every $u, v \in \mathcal{M}(X, \theta)$, $\partial(uv) = \partial(u)v + u\partial(v)$.*

Proof. Let $\pi\colon X^+ \to \mathcal{S}(X, \theta)$ (respectively, $\pi\colon X^* \to \mathcal{M}(X, \theta) = \mathcal{S}(X, \theta)^1$) be the canonical epimorphism. We also denote by $\pi\colon R\langle X\rangle^+ \to R\langle X, \theta\rangle^+$ (respectively, $\pi\colon R\langle X\rangle \to R\langle X, \theta\rangle$) the canonical epimorphism. (Recall that $R\langle X, \theta\rangle^+ = R[\mathcal{S}(X, \theta)] = R\langle X\rangle^+/\ker\pi$, respectively, $R\langle X, \theta\rangle = R[\mathcal{M}(X, \theta)] = R\langle X\rangle/\ker\pi$.) Let us choose a set-theoretic section $s\colon \mathcal{S}(X, \theta) \to X^+$ of π, *i.e.*, $\pi \circ s = id_{\mathcal{S}(X,\theta)}$ (respectively, $\pi \circ s = id_{\mathcal{M}(X,\theta)}$). It extends uniquely, as a R-linear map, to a section $s\colon R\langle X, \theta\rangle^+ \to R\langle X\rangle^+$ of π. Let us define $\partial_0\colon X \to R\langle X\rangle^+$ by $\partial_0(x) = s(\partial_X(x))$ for every $x \in X$. By Lemma 1, there exists a unique map $\partial_0\colon X^+ \to R\langle X\rangle^+$ that extends $\partial_0\colon X \to R\langle X\rangle^+$ such that for every $u, v \in X^+$, $\partial_0(uv) = \partial_0(u)v + u\partial_0(v)$. Now, let us check that $\pi \circ \partial_0\colon X^+ \to R\langle X, \theta\rangle^+$ passes to the quotient by $\ker\pi$. Let $(x, y) \in \theta$. We have $\partial_0(xy) = \partial_0(x)y + x\partial_0(y)$. Then,

$$\begin{aligned} \pi(\partial_0(xy)) &= \pi(\partial_0(x))\pi(y) + \pi(x)\pi(\partial_0(y)) \\ &= \pi(s(\partial_X(x)))\pi(y) + \pi(x)\pi(s(\partial_X(y))) \\ &= \partial_X(x)y + x\partial_X(y) \\ &= y\partial_X(x) + \partial_X(y)x \\ &\quad \text{(by assumption on } \partial_X) \\ &= \pi(\partial_0(yx)) \ . \end{aligned} \tag{2}$$

Let $w, w' \in X^+$ such that $\pi(w) = \pi(w')$. By definition of the congruence $\ker\pi$, there are $w = w_0, \ldots, w_n = w'$ such that for every $0 \le i < n$, $w_i = u_i x_i y_i v_i$, $w_{i+1} = u_i y_i x_i v_i$ such that $u_i, v_i \in X^*$ and $(x_i, y_i) \in \theta$. Then,

$$\begin{aligned}
\pi(\partial_0(w_i)) &= \pi(\partial_0(u_i x_i y_i v_i)) \\
&= \pi(\partial_0(u_i)x_i y_i v_i) + \pi(u_i)\pi(\partial_0(x_i))\pi(y_i)\pi(v_i) \\
&\quad +\pi(u_i)\pi(x_i)\pi(\partial_0(y_i))\pi(v_i) + \pi(u_i)\pi(x_i)\pi(y_i)\pi(\partial_0(v_i)) \\
&= \pi(\partial_0(u_i))x_i y_i \pi(v_i) + \pi(u_i)\partial_X(x_i)y_i\pi(v_i) \\
&\quad +\pi(u_i)x_i\partial_X(y_i)\pi(v_i) + \pi(u_i)x_i y_i \partial_0(v_i) \\
&= \pi(\partial_0(u_i))y_i x_i \pi(v_i) + \pi(u_i)y_i\partial_X(x_i)\pi(v_i) \\
&\quad +\pi(u_i)\partial_X(y_i)x_i\pi(v_i) + \pi(u_i)y_i x_i \partial_0(v_i) \\
&\quad \text{(since } (x_i, y_i) \in \theta \text{ and by assumption on } \partial_X) \\
&= \pi(\partial_0(w_{i+1}))\ .
\end{aligned} \tag{3}$$

It follows that $\pi(\partial_0(w)) = \pi(\partial_0(w_0)) = \cdots = \pi(\partial_0(w_n)) = \pi(\partial_0(w'))$. Therefore, there is a unique map $\partial\colon \mathcal{S}(X,\theta) \to R\langle X,\theta\rangle^+$ such that $\partial \circ \pi = \pi \circ \partial_0$. In particular, $\partial(\pi(x)) = \pi(\partial_0(x)) = \pi(s(\partial_X(x))) = \partial_X(x)$ for every $x \in X$. Moreover for every $u, v \in X^+$,

$$\begin{aligned}
\partial(\pi(u)\pi(v)) &= \partial(\pi(uv)) \\
&= \pi(\partial_0(uv)) \\
&= \pi(\partial_0(u)v + u\partial_0(v)) \\
&= \pi(\partial_0(u))\pi(v) + \pi(u)\pi(\partial_0(v)) \\
&= \partial(\pi(u))\pi(v) + \pi(u)\partial(\pi(v))\ .
\end{aligned} \tag{4}$$

It is easily checked that $\partial\colon \mathcal{S}(X,\theta) \to R\langle X,\theta\rangle^+$ is uniquely determined as a map that extends ∂_X and such that for every $u, v \in \mathcal{S}(X,\theta)$, $\partial(uv) = \partial(u)v + u\partial(v)$. Now, we extend uniquely ∂ to a map $\partial\colon \mathcal{M}(X,\theta) = \mathcal{S}(X,\theta) \sqcup \{\,\epsilon\,\} \to R\langle X,\theta\rangle$ by setting $\partial(\epsilon) = 0$. It is clear that for every $x \in X$, $\partial(x) = \partial_X(x)$, and for every $u, v \in \mathcal{M}(X,\theta)$, $\partial(uv) = \partial(u)v + u\partial(v)$. Moreover this is the unique extension of ∂ with this property. Furthermore, we already know that ∂ extends ∂_X and $\partial(uv) = \partial(u)v + u\partial(v)$ for every $u, v \in \mathcal{M}(X,\theta)$. This is the unique extension with such property. □

Lemma 3. *Let R be a ring, (X,θ) be any commutation alphabet, and let us give a set-theoretic map $\partial_X\colon X \to R\langle X,\theta\rangle^+$ such that for every $(x,y) \in \theta$, $\partial_X(x)y = y\partial_X(x)$. Then, there exists a unique derivation ∂ of $R\langle X,\theta\rangle^+$ such that for every $x \in X$, $\partial(x) = \partial_X(x)$. Moreover, ∂ extends uniquely to a derivation of $R\langle X,\theta\rangle$.This derivation may also be equivalently defined as the unique derivation of $R\langle X,\theta\rangle$ that extends ∂_X.*

Proof. According to Lemma 2, there are unique maps (which by abuse of language are denoted by the same name) $\partial\colon \mathcal{S}(X,\theta) \to R\langle X,\theta\rangle^+$ and $\partial\colon \mathcal{M}(X,\theta) \to R\langle X,\theta\rangle$ that extend ∂_X and such that for every $u, v \in \mathcal{S}(X,\theta)$ (respectively, $u, v \in \mathcal{M}(X,\theta)$), $\partial(uv) = \partial(u)v + u\partial(v)$. Since $R\langle X,\theta\rangle^+$ is free with basis $\mathcal{S}(X,\theta)$, then we may extend ∂ in a unique way as a R-linear map. Since $R\langle X,\theta\rangle$ is a free R-module with basis $\mathcal{M}(X,\theta) = \mathcal{S}(X,\theta) \sqcup \{\,\epsilon\,\}$, we may also extend ∂ in a unique way as a R-linear map. It is clear that this map $\partial\colon R\langle X,\theta\rangle \to R\langle X,\theta\rangle$ is an extension of the previous one $\partial\colon R\langle X,\theta\rangle^+ \to R\langle X,\theta\rangle^+$, and thus an extension of ∂_X. Both linear maps are actually derivations as it is easy to check. Uniqueness of both maps is obvious. □

4.3 Non-commutative Case

In this subsection is defined the differential algebra of a semigroup (respectively, monoid) S. We denote by $\times$ the product in S, and its identity by 1_S. In what follows, for every set Y, we denote by X_Y the set $Y \times \mathbb{N}$.

According to Lemmas 1 and 3 (with $\theta = \emptyset$), we define in a unique way a R-derivation ∂ on $R\langle X_S\rangle^+$ (respectively, $R\langle X_S\rangle$) by setting $\partial(x^{(i)}) = x^{(i+1)}$ for every $(x,i) \in S \times \mathbb{N}$.

Let us consider the relations on $R\langle X_S\rangle^+$ (respectively, $R\langle X_S\rangle$): for every $i \geq 0$ and every $x, y \in S$, $(x \times y)^{(i)} = \sum_{j=0}^{i} \binom{i}{j} x^{(j)} y^{(i-j)}$. (Respectively, in addition with the relations $1_S^{(0)} = \epsilon$, and $1_S^{(i)} = 0$ for every $i > 0$.) In particular, $(x \times y)^{(0)} = x^{(0)}y^{(0)}$ for every $x, y \in S$.

Let us denote by $R\wr S\int$ the quotient R-algebra $R\langle X_S\rangle^+$ (respectively, unital R-algebra $R\langle X_S\rangle$) by the two-sided ideal I generated by the above relations. The corresponding congruence is denoted by $\equiv$, and π is the canonical epimorphism from $R\langle X_S\rangle^+$ (respectively, from $R\langle X_S\rangle$) onto $R\wr S\int = R\langle X_S\rangle^+/_{\equiv}$ (respectively, $R\wr S\int = R\langle X_S\rangle/_{\equiv}$). Let $x, y \in S$ and $i \in \mathbb{N}$. We have $\partial((x \times y)^{(i)}) = (x \times y)^{(i+1)}$, while we have $\partial\left(\sum_{j=0}^{i} \binom{i}{j} x^{(j)} y^{(i-j)}\right) = \sum_{j=0}^{i} \binom{i}{j} \partial(x^{(j)} y^{(i-j)}) = \sum_{j=0}^{i+1} \binom{i+1}{j} x^{(j)} y^{(i+1-j)}$. Therefore, $\partial((x \times y)^{(i)}) \equiv \partial\left(\sum_{j=0}^{i} \binom{i}{j} x^{(j)} y^{(i-j)}\right)$. (If S is a monoid, we also have $\partial(1^{(0)}) = 1^{(1)} \equiv 0 = \partial(\epsilon)$ and for every $i > 0$, $\partial(1^{(i)}) = 1^{(i+1)} \equiv 0 = \partial(0)$.) From these results, it is easy to check that I is a differential ideal. So, we can define in a unique way a R-linear derivation $\overline{\partial}\colon R\wr S\int \to R\wr S\int$ such that $\overline{\partial}(\pi(p)) = \pi(\partial(p))$ for every $p \in R\langle X_S\rangle^+$ (respectively, $p \in R\langle X_S\rangle$). The map $q_S\colon x \in S \mapsto \pi(x^{(0)}) \in R\wr S\int$ is a homomorphism of semigroups (respectively, monoids). Indeed, $\pi((x \times y)^{(0)}) = \pi(x^{(0)}y^{(0)}) = \pi(x^{(0)})\pi(y^{(0)})$ (plus $\pi(1_S^{(0)}) = \pi(\epsilon)$ if S is a monoid). The algebra $R\wr S\int$ satisfies a universal property as it is stated in the following lemma.

Lemma 4. *Let S be any semigroup (respectively, a monoid), and (A, ∂_A) be a R-differential algebra (respectively, unital R-algebra). Let $\phi\colon S \to A$ be a homomorphism of semigroups (respectively, monoids) from S to the multiplicative structure of A. Then, there is a unique homomorphism of differential R-algebras (respectively, unital differential R-algebras) $\widehat{\phi}\colon (R\wr S\int, \overline{\partial}) \to (A, \partial_A)$ such that $\widehat{\phi}(q_S(x)) = \phi(x)$ for every $x \in S$.*

Proof. We define $\widehat{\phi}\colon (R\wr S\int, \overline{\partial}) \to (A, \partial_A)$ in several steps. Let $\phi_1\colon X_S \to A$ be defined by $\phi_1(x^{(i)}) = \partial_A^i(\phi(x))$ for every $(x,i) \in S \times \mathbb{N}$. In particular, $\phi_1(x^{(0)}) = \partial_A^0(\phi(x)) = \phi(x)$. Then, we define $\phi_2\colon X_S^+ \to A$ (respectively, $\phi_2\colon X_S^* \to A$) as the unique semigroup (respectively, monoid) homomorphism extension of ϕ_1. That is, $\phi_2(x^{(i)}) = \phi_1(x^{(i)}) = \partial_A^i(\phi(x))$ for every $(x,i) \in S \times \mathbb{N}$, and $\phi_2(a_1 \cdots a_n) = \phi_1(a_1) \cdots \phi_1(a_n)$ for every $a_k \in S \times \mathbb{N}$ (plus, $\phi_2(\epsilon) = 1_A$ if S is a monoid). Now, $\phi_3\colon R\langle X_S\rangle^+ \to A$ (respectively, $\phi_3\colon R\langle X_S\rangle \to A$) is the unique extension of ϕ_2 as the homomorphism of algebras (respectively, unital algebras). This is possible since $R\langle X_S\rangle^+ = R[X_S^+]$ (respectively, $R\langle X_S\rangle = R[X_S^*]$). Let us prove that ϕ_3 passes to the quotient to $R\wr S\int$. Let $i \in \mathbb{N}$, $x, y \in S$. We have

$\phi_3((x\times y)^{(i)})=\phi_2((x\times y)^{(i)})=\phi_1((x\times y)^{(i)})=\partial_A^i(\phi(x\times y))=\partial_A^i(\phi(x)\phi(y))$, and

$$\begin{aligned}\phi_3\left(\sum_{j=0}^i\binom{i}{j}x^{(j)}y^{(i-j)}\right)&=\sum_{j=0}^i\binom{i}{j}\phi_3(x^{(j)}y^{(i-j)}))\\&=\sum_{j=0}^i\binom{i}{j}\phi_3(x^{(j)})\phi_3(y^{(i-j)}))\\&=\sum_{j=0}^i\binom{i}{j}\phi_1(x^{(j)})\phi_1(y^{(i-j)}))\\&=\sum_{j=0}^i\binom{i}{j}\partial_A^j(\phi(x))\partial_A^{i-j}(\phi(y))\\&=\partial_A^i(\phi(x)\phi(y))\end{aligned}\tag{5}$$

(and $\phi_3(1_S^{(0)})=\phi(1_S)=1_A=\phi_3(\epsilon)$, $\phi_3(1_S^{(i)})=\partial_A^i(\phi(1_S))=\partial_A^i(1_A)=0_A$, $\phi_3(0)=0_A$ for every $i>0$, if S is a monoid). Therefore, we define a homomorphism of algebras (respectively, unital algebras) in a unique way $\widehat{\phi}\colon R\wr S\wr\to A$ by setting $\widehat{\phi}(\pi(p))=\phi_3(p)$ for every $p\in R\langle X_S\rangle^+$ (respectively, $p\in R\langle X_S\rangle$). In particular, $\widehat{\phi}(\pi(x^{(0)}))=\phi_3(x^{(0)})=\phi_2(x^{(0)})=\phi_1(x^{(0)})=\partial_A^0(\phi(x))=\phi(x)$ for every $x\in S$. Let us check that $\widehat{\phi}$ commutes with the derivations. First, let $w\in X_S^+$, and let us prove that $\phi_3(\partial(w))=\partial_A(\phi_2(w))$ by induction. First we have $\phi_3(\partial(x^{(i)}))=\phi_3(x^{(i+1)})=\phi_1(x^{(i+1)})=\partial_A^{i+1}(\phi(x))=\partial_A\circ\partial_A^i(\phi(x))=\partial_A(\phi_1(x^{(i)}))=\partial_A(\phi_3(x^{(i)}))$ for every $x\in S$, $i\in\mathbb{N}$. We have

$$\begin{aligned}\phi_3(\partial(x^{(i)}u))&=\phi_3(x^{(i+1)}u+x^{(i)}\partial(u))\\&=\phi_3(x^{(i+1)}u)+\phi_3(x^{(i)}\partial u)\\&=\phi_3(x^{(i+1)})\phi_3(u)+\phi_3(x^{(i)})\phi_3(\partial(u))\\&=\phi_1(x^{(i+1)})\phi_3(u)+\phi_1(x^{(i)})\partial_A(\phi_3(u))\\&\quad\text{(by induction)}\\&=\partial_A^{i+1}(\phi(x))\phi_3(u)+\partial_A^i(\phi(x))\partial_A(\phi_3(u))\\&=\partial_A(\partial_A^i(\phi(x))\phi_3(u))\\&=\partial_A(\phi_1(x^{(i)})\phi_3(u))\\&=\partial_A(\phi_3(x^{(i)})\phi_3(u))\\&=\partial_A(\phi_3(x^{(i)}u))\\&=\partial_A(\phi_2(x^{(i)}u))\end{aligned}\tag{6}$$

for every $(i,x,u)\in\mathbb{N}\times S\times X_S^+$ (moreover, if S is a monoid, then $\phi_3(\partial(\epsilon))=\phi_3(0)=0_A=\partial_A(0_A)=\partial_A(\phi_3(\epsilon))$). Now, let $p=\sum_{w\in X_S^+}p_w w\in R\langle X_S\rangle^+$. We have

$$\begin{aligned}\phi_3(\partial p)&=\phi_3\left(\sum_{w\in X_S^+}p_w\partial(w)\right)\\&=\sum_{w\in X_S^+}p_w\phi_3(\partial(w))\\&=\sum_{w\in X_S^+}p_w\partial_A(\phi_2(w))\\&=\partial_A(\phi_3(p))\ .\end{aligned}\tag{7}$$

Finally, we have $\widehat{\phi}(\overline{\partial}(\pi(p)))=\widehat{\phi}(\pi(\partial(p)))=\phi_3(\partial(p))=\partial_A(\phi_3(p))=\partial_A(\widehat{\phi}(\pi(p)))$, which proves that $\widehat{\phi}$ commutes with the derivations (the same proof holds for $p\in R\langle X_S\rangle$). It remains to check that $\widehat{\phi}$ is the unique homomorphism of differential R-algebras (respectively, unital R-differential algebras) such that $\widehat{\phi}(\pi(x^{(0)}))=\phi(x)$ for every $x\in S$. But this is quite easy. □

Lemma 4 shows some similarities between $R[S]$ and $R\lbag S\rbag$. This is the reason why the differential R-algebra $(R\lbag S\rbag, \overline{\partial})$ is called the *differential R-algebra of the semigroup* (respectively, *monoid*) S. We now give a relation between the differential algebra of a semigroup S and that of the monoid S^1 obtained by free adjunction of an identity to S. Recall from Subsection 4.1 that if A is a R-algebra, then $A_1 = A \oplus R$ denotes the (universal) unital R-algebra generated by A.

Lemma 5. *If (A, ∂_A) is a differential R-algebra, then (A_1, ∂_{A_1}) is a differential R-algebra with $\partial_{A_1}(x, \alpha) = (\partial_A(x), 0)$, $x \in A$. More precisely it is universal in the following sense: for every unital differential R-algebra (B, ∂_B), and every homomorphism $\phi\colon (A, \partial_A) \to (B, \partial_B)$ of differential R-algebras, there is a unique homomorphism $\phi_1\colon (A_1, \partial_{A_1}) \to (B, \partial_B)$ of unital differential R-algebras such that $\phi_1(x, 0) = \phi(x)$ for every $x \in A$.*

Proof. Let $\partial_{A_1}\colon A_1 \to A_1$ be defined by $\partial_{A_1}(x, \alpha) = (\partial_A(x), 0)$ for every $x \in A$, $\alpha \in R$. It is easily checked that it is a R-linear derivation on A_1. Now, let $\phi\colon (A, \partial_A) \to (B, \partial_B)$ be a homomorphism of differential R-algebras, where (B, ∂_B) is a unital differential R-algebra. Let $\phi_1\colon A_1 \to B$ be the unique homomorphism of unital algebras that extends ϕ, *i.e.*, $\phi_1(x, \alpha) = \phi(x) + \alpha 1_B$ for all $x \in A$, $\alpha \in R$, also is a homomorphism of unital differential R-algebras. Indeed, let $(x, \alpha) \in A_1$. We have $\phi_1(\partial_{A_1}(x, \alpha)) = \phi_1(\partial_A(x), 0) = \phi(\partial_A(x)) = \partial_B(\phi(x)) = \partial_B(\phi(x) + \alpha 1_B) = \partial_B(\phi_1(x, \alpha))$. Uniqueness is obvious. □

Remark 2. If we consider members of $A_1 = A \oplus R$ as sums $x + \alpha$, $x \in A$, $\alpha \in R$, then $\partial_{A_1}(x + \alpha) = \partial_A(x)$ for every $x \in A$.

If S is a semigroup, then as presented in Subsection 4.1, $S^1 = S \sqcup \{1\}$ denotes the monoid obtained by adjunction of a unit to S.

Lemma 6. *Let S be a semigroup. We have $(R\lbag S^1\rbag, \overline{\partial}) \cong (R\lbag S\rbag_1, \overline{\partial})$ as unital R-differential algebras.*

Proof. Let $\phi\colon S^1 \to A$ be a monoid homomorphism from S^1 to the multiplicative monoid of a unital algebra A, where (A, ∂_A) is a unital differential R-algebra. There exists a unique homomorphism of semigroups $\phi_0\colon S \to A$ such that $\phi(1_S) = 1_A$ and $\phi(s) = \phi_0(s)$ for every $s \in S$. According to Lemma 4, there is a unique homomorphism of differential R-algebras $\widehat{\phi}_0\colon (R\lbag S\rbag, \overline{\partial}) \to (A, \partial_A)$ such that $\widehat{\phi}_0(\pi(x^{(0)})) = \phi_0(x) = \phi(x)$. According to Lemma 5, there exists a unique homomorphism of unital differential R-algebra $\widehat{\phi}_{0,1}\colon (R\lbag S\rbag_1, \overline{\partial}_1) \to (A, \partial_A)$ such that $\widehat{\phi}_{0,1}(x, 0) = \widehat{\phi}_0(x)$ for every $x \in R\lbag S\rbag$. Since $\widehat{\phi}_{0,1}(0, 1) = 1_A = \phi(1)$, this implies that $(R\lbag S\rbag_1, \overline{\partial}_1)$ satisfies the same universal problem as $(R\lbag S^1\rbag, \overline{\partial})$. □

4.4 Commutative Case

The results from Subsection 4.3 are extended to commutative semigroups and monoids. Recall from Subsection 4.1 that $\mathcal{S}(X)$ (respectively, $\mathcal{M}(X)$) denotes the free commutative semigroup (respectively, monoid), *i.e.*, the free partially commutative semigroup (respectively, monoid) $\mathcal{S}(X, \Delta_X)$ (respectively, $\mathcal{M}(X, \Delta_X)$) where $\Delta_X = \{ (x, y) \in X^2 \colon x \neq y \}$.

Let S be a commutative semigroup (respectively, monoid). According to Lemmas 2 and 3, we define in a unique way a R-derivation ∂ on $R[X_S]^+$ (respectively, $R[X_S]$), where $X_S = S \times \mathbb{N}$, by setting $\partial(x^{(i)}) = x^{(i+1)}$ for every $(x, i) \in S \times \mathbb{N}$ (since $R[X_S]^+ = R\langle X_S, \Delta_{X_S}\rangle^+$, and respectively, $R[X_S] = R\langle X_S, \Delta_{X_S}\rangle$). Let us consider the relations on $R[X_S]^+$ (respectively, $R[X_S]$): for every $i \geq 0$ and every $x, y \in S$, $(x \times y)^{(i)} = \sum_{j=0}^{i} \binom{i}{j} x^{(j)} y^{(i-j)}$. (And in addition, the relations $1_S^{(0)} = \epsilon$, and $1_S^{(i)} = 0$ for every $i > 0$, if S is a monoid.) In particular, $(x \times y)^{(0)} = x^{(0)} y^{(0)}$ for every $x, y \in S$. Let us denote by $R\{S\}$ the quotient R-algebra $R[X_S]^+$ (respectively, unital R-algebra $R[X_S]$) by the ideal I generated by the above relations. The corresponding congruence is denoted by $\equiv$, and π is the canonical epimorphism from $R[X_S]^+$ (respectively, from $R[X_S]$) onto $R\{S\} = R[X_S]^+/_{\equiv}$ (respectively, $R\{S\} = R[X_S]/_{\equiv}$). It is easy to check that $\partial(I) \subseteq I$ so that I is a differential ideal. So we can define in a unique way a R-linear derivation $\overline{\partial} \colon R\{S\} \to R\{S\}$ such that $\overline{\partial}(\pi(p)) = \pi(\partial(p))$ for every $p \in R[X_S]^+$ (respectively, $p \in R[X_S]$). The map $q_S \colon x \in S \mapsto \pi(x^{(0)}) \in R\{S\}$ is a homomorphism of commutative semigroups (respectively, monoids). Indeed, $\pi((x \times y)^{(0)}) = \pi(x^{(0)} y^{(0)}) = \pi(x^{(0)})\pi(y^{(0)})$ (and $\pi((1_S^{(0)})) = \pi(\epsilon)$, if S is a commutative monoid).

Lemma 7. *Let S be any commutative semigroup (respectively, monoid), and (A, ∂_A) be a R-differential commutative algebra (respectively, unital commutative R-algebra). Let $\phi \colon S \to A$ be a homomorphism of commutative semigroups (respectively, monoids) from S to the multiplicative structure of A. Then, there is a unique homomorphism of differential commutative R-algebras (respectively, unital differential commutative R-algebras) $\widehat{\phi} \colon (R\{S\}, \overline{\partial}) \to (A, \partial_A)$ such that $\widehat{\phi}(\pi(x^{(0)})) = \phi(x)$ for every $x \in S$.*

Proof. The proof is omitted since it is almost only an adaptation of that of Lemma 4. □

The differential R-algebra $(R\{S\}, \overline{\partial})$ is called the *commutative differential R-algebra of the (commutative) semigroup* (respectively, *monoid*) S. As in the non-commutative case, there is a relation between $R\{S^1\}$ and $R\{S\}_1$ given by Lemma 9 and which used the following result (as easily proved as Lemma 5).

Lemma 8. *If (A, ∂_A) is a commutative differential R-algebra, then (A_1, ∂_{A_1}) is a commutative differential R-algebra with $\partial_{A_1}(x, \alpha) = (\partial_A(x), 0)$ for all $x \in A$. More precisely it is universal in the following sense: for every commutative unital differential R-algebra (B, ∂_B), and every homomorphism $\phi \colon (A, \partial_A) \to (B, \partial_B)$ of commutative differential R-algebras, there is a unique homomorphism $\phi_1 \colon (A_1, \partial_{A_1}) \to (B, \partial_B)$ of commutative unital differential R-algebras such that $\phi_1(x, 0) = \phi(x)$ for every $x \in A$.*

The following lemma is then easily proved.

Lemma 9. *Let S be a commutative semigroup. We have $(R\{S^1\}, \overline{\partial}) \cong (R\{S\}_1, \overline{\partial}_1)$ as commutative unital R-differential algebras.*

4.5 Abelianization of Differential Algebras

There is an obvious forgetful functor $R\text{-}\mathcal{CDiffAlg} \to R\text{-}\mathcal{DiffAlg}$ (respectively, a forgetful functor $R\text{-}\mathcal{CDiffAlg}_1 \to R\text{-}\mathcal{DiffAlg}_1$) that admits a left adjoint, the well-known abelianization functor (in the case of algebras without derivation) that we describe now.

It is well-known that the forgetful functor from commutative semigroups (respectively, commutative monoids, commutative R-algebras, commutative and unital R-algebras) to semigroups (respectively, monoids, R-algebras, unital R-algebras) has a left adjoint, also called the abelianization functor. It is defined as follows: let S be either a semigroup, a monoid, a R-algebra, or a unital R-algebra. Let $\mathcal{Ab}(S)$ be the quotient semigroup (respectively, monoid, R-algebra, unital R-algebra) by the least congruence generated by the relations $xy = yx$ for every $x, y \in S$ (respectively, by the two-sided ideal I generated by all the commutators $xy - yx$ in the cases of algebras). Let (A, ∂_A) be a differential R-algebra (respectively, unital differential R-algebra). It is easy to see that I is actually a differential ideal. Therefore, $\mathcal{Ab}(A)$ admits a natural structure of a commutative (respectively, unital and commutative) differential R-algebra: let $\pi_{\mathcal{Ab}} \colon A \to \mathcal{Ab}(A)$ be the canonical epimorphism from A to its abelianization. Then, there is a unique derivation $\partial_{\mathcal{Ab}(A)}$ on $\mathcal{Ab}(A)$ such that $\partial_{\mathcal{Ab}(A)} \circ \pi_{\mathcal{Ab}} = \pi_{\mathcal{Ab}} \circ \partial_A$ and $\pi_{\mathcal{Ab}}$ becomes a homomorphism of differential R-algebras (respectively, commutative differential R-algebras).

Lemma 10. *Let (A, ∂_A) be a differential R-algebra (respectively, a unital differential R-algebra), then its abelianization is $(\mathcal{Ab}(A), \partial_{\mathcal{Ab}(A)})$.*

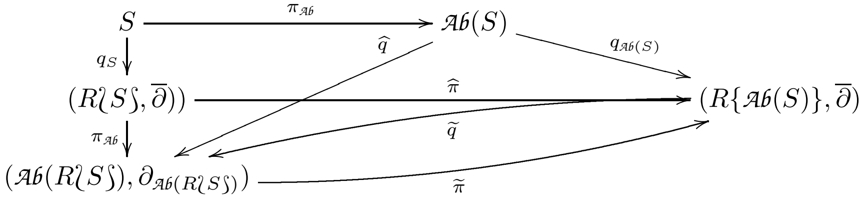
Proof. Let $\phi \colon (A, \partial_A) \to (B, \partial_B)$ be a homomorphism of R-algebras (respectively, unital R-algebras), where (B, ∂_B) is assumed to be a commutative (respectively, unital and commutative) R-algebra. Let $\widetilde{\phi} \colon \mathcal{Ab}(A) \to B$ be the corresponding abelianization of ϕ for the underlying R-algebras (respectively, unital R-algebras). Let us check that $\widetilde{\phi} \circ \partial_{\mathcal{Ab}(A)} = \partial_B \circ \widetilde{\phi}$. We have

$$\begin{aligned} \widetilde{\phi} \circ \partial_{\mathcal{Ab}(A)} \circ \pi_{\mathcal{Ab}} &= \widetilde{\phi} \circ \pi_{\mathcal{Ab}} \circ \partial_A \\ &= \phi \circ \partial_A \\ &= \partial_B \circ \phi \\ &= \partial_B \circ \widetilde{\phi} \circ \pi_{\mathcal{Ab}} \ . \end{aligned} \tag{8}$$

Let $\psi \colon (\mathcal{Ab}(A), \partial_{\mathcal{Ab}(A)}) \to (B, \partial_B)$ be a homomorphism of commutative (respectively, unital and commutative) differential R-algebras such that $\psi \circ \pi_{\mathcal{Ab}} = \phi$. It is therefore clear that $\widetilde{\phi} = \psi$ as homomorphisms of commutative R-algebras, and so as homomorphisms of commutative differential R-algebras. □

Lemma 11. *Let S be a semigroup (respectively, monoid). Then, as commutative differential R-algebras $(\mathcal{Ab}(R\wr S \wr), \partial_{\mathcal{Ab}(R\wr S\wr)}) \cong (R\{\mathcal{Ab}(S)\}, \overline{\partial})$ (respectively, as* *commutative and unital differential R-algebras, we have* $(\mathcal{A}b(R\wr M\wr), \partial_{\mathcal{A}b(R\wr M\wr)}) \cong (R\{\mathcal{A}b(M)\}, \overline{\partial})$.

Proof. According to the universal problems, we have the following commutative diagram.



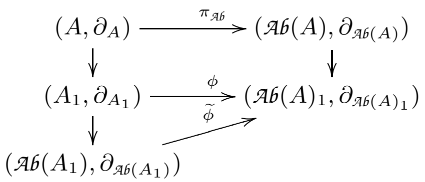
(9)

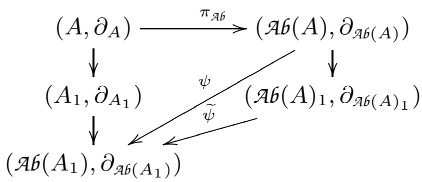
That is, there are natural arrows such that $\widehat{\pi} \circ q_S = q_{\mathcal{A}b(S)} \circ \pi_{\mathcal{A}b}$, $\widetilde{\pi} \circ \pi_{\mathcal{A}b} = \widehat{\pi}$, $\widetilde{q} \circ q_{\mathcal{A}b(S)} = \widehat{q}$ and $\widehat{q} \circ \pi_{\mathcal{A}b} = \pi_{\mathcal{A}b} \circ q_S$. Therefore, $\widetilde{\pi} \circ \widetilde{q} \circ q_{\mathcal{A}b(S)} \circ \pi_{\mathcal{A}b} = \widetilde{\pi} \circ \pi_{\mathcal{A}b} \circ q_S = q_{\mathcal{A}b(S)} \circ \pi_{\mathcal{A}b}$, so that $\widetilde{\pi} \circ \widetilde{q} = id_{(R\{\mathcal{A}b(S)\}, \overline{\partial})}$, and $\widetilde{q} \circ \widetilde{\pi} \circ \pi_{\mathcal{A}b} \circ q_S = \widetilde{q} \circ q_{\mathcal{A}b(S)} \circ \pi_{\mathcal{A}b} = \pi_{\mathcal{A}b} \circ q_S$, so that $\widetilde{q} \circ \widetilde{\pi} = id_{(\mathcal{A}b(R\wr S\wr), \partial_{\mathcal{A}b(R\wr S\wr)})}$ (by usual categorical arguments). □

It is also possible to present a connection between abelianization and unitarization as follows.

Lemma 12. *Let* (A, ∂_A) *be a R-differential algebra. Then,* $(\mathcal{A}b(A_1), \partial_{\mathcal{A}b(A_1)}) \cong (\mathcal{A}b(A)_1, \partial_{\mathcal{A}b(A)_1})$ *as commutative unital R-algebras.*

Proof. The proof is due to the following commutative diagrams





where the arrows without names are the canonical arrows. □

In particular, for any semigroup S, $R\{\mathcal{A}b(S^1)\}$, $R\{\mathcal{A}b(S)^1\}$, $R\{\mathcal{A}b(S)\}_1$, $\mathcal{A}b(R\wr S^1\wr)$, $\mathcal{A}b(R\wr S\wr_1)$ and $\mathcal{A}b(R\wr S\wr)_1$ are all naturally isomorphic as commutative unital differential algebras.

4.6 Embedding of S into $R\wr S\wr$

Let S be a semigroup (respectively, commutative semigroup, monoid, commutative monoid), and $R[S]$ be the usual R-algebra of S. Let $j_S : S \to R[S]$ be the natural inclusion which is a homomorphism of semigroups (respectively, monoids), and $q_S : S \to R\{S\}$ be the homomorphism, already defined in Subsection 4.4, such that $q_S(x) = \pi(x^{(0)})$ for every $x \in S$. Since $(R[S], 0)$ is also a differential R-algebra (respectively, commutative differential R-algebra, if S

is commutative, and unital differential R-algebra, if S has an identity), according to Lemma 4, there is a unique homomorphism of differential R-algebras $\widehat{j}_S\colon (R\wr S\wr, \overline{\partial}) \to (R[S], 0)$ (respectively, by Lemma 7, if S is commutative, a unique homomorphism of commutative differential R-algebras, $\widehat{j}_S\colon (R\{S\}, \overline{\partial}) \to (R[S], 0)$) such that $\widehat{j}_S \circ q_S = j_S$. This implies immediatly that q_S is one-to-one so that S embeds into $R\wr S\wr$ (respectively, for the commutative case, into $R\{S\}$) as a semigroup (respectively, monoid).

In Section 5 is defined the differential algebra generated by a (usual) algebra and is proved that the later is embedded (as a sub-algebra) into the former. This may be used to prove the embedding of $R[S]$ into $R\wr S\wr$ (respectively, $R\{S\}$, depending on whether or not S is commutative). More precisely, see remarks 5 and 7.

Lemma 13. *The functor $J\colon R\text{-}\mathcal{Alg} \to R\text{-}\mathcal{DiffAlg}$ (respectively, $R\text{-}\mathcal{CAlg} \to R\text{-}\mathcal{CDiffAlg}$, $R\text{-}\mathcal{Alg}_1 \to R\text{-}\mathcal{DiffAlg}_1$, $R\text{-}\mathcal{CAlg}_1 \to R\text{-}\mathcal{CDiffAlg}_1$), that maps a R-algebra A to the differential R-algebra $(A, 0)$ and which is the identity at the level of homomorphisms, is a full embedding.*

Proof. We prove this lemma only for the first case. The other cases are treated similarly. Let A, B be two R-algebras. If $(A, 0) = (B, 0)$, then it is clear that $A = B$ so that J is injective on objects. It is faithful: let A be a R-algebra. Let f, g be two homomorphisms from A to B such that $J(f) = J(g)$. Then, it is clear that $f = g$. Let A, B be two R-algebras and let $g\colon (A, 0) \to (B, 0)$ be a homomorphism. Then, $g\colon A \to B$ is a usual homomorphism, and $J(g) = g$ so that J is full. □

According to Lemma 13, $R\text{-}\mathcal{Alg}$ (respectively, $R\text{-}\mathcal{Alg}_1$, respectively, $R\text{-}\mathcal{CAlg}$, respectively, $R\text{-}\mathcal{CAlg}_1$) may be identified as a full subcategory of $R\text{-}\mathcal{DiffAlg}$ (respectively, $R\text{-}\mathcal{DiffAlg}_1$, respectively, $R\text{-}\mathcal{CDiffAlg}$, respectively, $R\text{-}\mathcal{CDiffAlg}_1$), namely that of algebras with a zero derivation. It becomes clear that $(R[S], 0)$ for a semigroup (respectively, a monoid) S is the free differential semigroup (respectively, monoid) algebra generated by S in the full subcategory of all differential algebras with the zero derivation (and the same holds for the commutative case).

4.7 A Transversal for $R\wr A^+\wr$

In this subsection is studied the differential algebra $R\wr A^+\wr$ generated by the free semigroup A^+ on the set A. The objective is here to provide a normal form for the elements of this quotient algebra. In other terms we construct a transversal (see [6]) for the equivalence relation $\equiv$ such that $R\langle X_{A^+}\rangle^+/_{\equiv} \cong R\wr A^+\wr$ (see Subsection 4.3), *i.e.*, a subset T of $R\langle X_{A^+}\rangle^+$ that meets each equivalence classes modulo $\equiv$ in exactly one element. This is equivalent to find a section $s\colon R\wr A^+\wr \hookrightarrow R\langle X_{A^+}\rangle^+$ to the natural epimorphism $\pi\colon R\langle X_{A^+}\rangle^+ \to R\wr A^+\wr$. Actually we will prove that $R\wr A^+\wr$ is isomorphic to $R\langle A \times \mathbb{N}\rangle^+$ (and thus, $R\wr A^*\wr$ is isomorphic to $R\langle A \times \mathbb{N}\rangle$). In order to provide this result, we use the theory of polynomials reduction systems of G.M. Bergman in [1].

Let S be a semigroup (respectively, monoid), with product $\times$ (and identity 1_S if S is a monoid). Recall that $X_S = S \times \mathbb{N}$. We consider the following *reduction system* (see [1]) on $R\langle X_S\rangle^+$, $\mathcal{Red}_S = \{\, ((x \times y)^{(i)}, \sum_{j=0}^{i} \binom{i}{j} x^{(j)} y^{(i-j)}) \colon i \geq 0,\ x, y \in S \,\}$. (Of which are added the following rules $\{(1_S^{(0)}, \epsilon)\,\} \sqcup \{\,(1_S^{(i)}, 0) \colon i > 0\,\}$ if S is a monoid.) Following the terminology from [1], there are no overlap ambiguities. But there are inclusion ambiguities which may be not resolvable. Indeed, let us assume that $w \times x = y \times z$ in S, then we may reduce $(w \times x)^{(i)}$ as $\sum_{j=0}^{i} \binom{i}{j} w^{(j)} x^{(j-i)}$ and $(w \times x)^{(i)} = (y \times z)^{(i)}$ as $\sum_{j=0}^{i} \binom{i}{j} y^{(j)} z^{(j-i)}$. But there may be no ways to reduce both polynomials to a same one.

Let us assume for a time that S is the free semigroup A^+ over some set A. Then, every (inclusion) ambiguity is resolvable. Let $|w|$ be the length of any word w in A^+. Let us assume that $u_1u_2 = v_1v_2$ in A^+. If $|u_1| = |v_1|$, then $u_1 = v_1$, and $u_2 = v_2$, so that the ambiguity is obviously resolved. So let us assume for instance that $|u_1| < |v_1|$, then $|u_2| > |v_2|$. There exists w such that $v_1 = u_1w$, and $u_2 = wv_2$. We have $(u_1u_2)^{(i)} = (u_1(wv_2))^{(i)}$ which reduces to $\sum_{j=0}^{i} \binom{i}{j} (u_1)^{(j)} (wv_2)^{(i-j)}$, while $(v_1v_2)^{(i)} = ((u_1w)v_2)^{(i)}$ reduces to $\sum_{j=0}^{i} \binom{i}{j} (u_1w)^{(j)} (v_2)^{(i-j)}$. By associativity in X_S^+, both sums are joinable (by reduction applied on each of them).

Now, let us assume that S is any semigroup. We define the following relations on X_S^+: $u(x \times y)^{(i)}v \to ux^{(j)}y^{(i-j)}v$ for every $i \geq 0$, $0 \leq j \leq i$, $x, y \in S$ and $u, v \in X_S^*$. It is an irreflexive relation since $|w| < |w'|$ for every w, w' such that $w \to w'$. We now consider the transitive closure $\to^+$ of $\to$ defined by $w \to^+ w'$ if, and only if, there are $w_0 = w, w_1, \ldots, w_n = w'$, $n > 0$, such that $w_i \to w_{i+1}$ for $0 \leq i \leq n-1$. This relation is also obviously irreflexive since $|w| < |w'|$ whenever $w \to^+ w'$. Therefore, $\to^+$ is a strict (partial) order on X_S^+. We define $w \Rightarrow^+ w'$ by $w \to^+ w'$ or $w = w'$. This defines a partial order on X_S^+. It satisfies the following: let us assume that $w \to^+ w'$, and let $u, v \in X_S^*$, then $uwv \to^+ uw'v$, and therefore it is called a *semigroup partial ordering* according to [1]. Since for every $i \geq 0$, and every $x, y \in S$, $(x \times y)^{(i)} \to^+ x^{(j)}y^{(j-i)}$, the partial order $\Rightarrow^+$ is said to be *compatible* with the reduction system. An *irreducible monomial* (under $\mathcal{Red}_S$) is a word $x_1^{(i_1)} \cdots x_n^{(i_n)} \in X_S^+$ such that for every $1 \leq j \leq n$, there are no $y, z \in S$ such that $x_j = y \times z$ in S, $j = 1, \ldots, n$. The set of all irreducible monomials under $\mathcal{Red}_S$ is denoted by $\mathcal{Irr}_S$. An *irreducible polynomial* is a member of $R\langle X_S\rangle^+$ that involves only irreducible monomials. Let $x \in S$.

A *decomposition of length* n of x is a the sequence $(x_1, \ldots, x_n) \in S^n$ such that $x_1 \times \cdots \times x_n = x$ in S. The set of decompositions of length n of x is denoted by $D_n(x)$. A semigroup is said to be *locally finite* if for every $x \in S$, then $\bigcup_{n \geq 0} D_n(x)$ is finite (see [11]).

Lemma 14. *Assuming that S is locally finite, the partial order $\Rightarrow^+$ has descending chain condition.*

Proof. Let $(u_n)_{n \geq 0}$ be a sequence of member of X_S^+ such that for every $n \geq 0$, $u_n \to^+ u_{n+1}$. By definition of $\to^+$, we may assume without loss of generality that $u_n \to u_{n+1}$ for every n. Let us assume that $u_0 = x_{0,1}^{(i_{0,1})} \cdots x_{0,n_0}^{(i_{0,n_0})}$, $n_0 \geq 1$,

$x_{0,j} \in S$, $i_{0,n_j} \in \mathbb{N}$, for $j = 0, \ldots, n_0$. Let $x_0 = x_{0,1} \times \cdots \times x_{0,n_0}$ so that $(x_{0,1}, \ldots, x_{0,n_0}) \in D_{n_0}(x_0)$. By induction it follows that $|u_n| = n_0 + n$ for each n. Therefore $u_n = x_{n,1}^{(i_{n,1})} \cdots x_{n,n_0+n}^{(i_{n,n_0+n})}$, with $x_{n,k} \in S$ for every $k = 1, \ldots, n_0 + n$. Moreover, again by induction, we have $x_0 = x_{n,1} \times \cdots \times x_{n,n_0+n}$ so that $D_{n_0+n}(x_0) \neq \emptyset$ for every n which contradict local finiteness of S. □

Remark 3. The monoid case is fundamentally different since there are infinite decreasing reductions even for the trivial monoid: indeed, let x be a member of a monoid M, then $x^{(0)} = (x \times 1_M)^{(0)} \to x^{(0)}(1_M)^{(0)} \to x^{(0)} \to \cdots$

Now, if S is the free semigroup A^+, then it is locally finite, and $S \setminus S^{(2)} = A$ (where $S^{(2)} = \{\, x \in S \colon D_2(x) \neq \emptyset \,\}$). Moreover we already know that in this case all inclusion ambiguities are resolvable. Then, with Lemma 14, it allows us to apply Theorem 1.2 from [1] (easily generalized to the free algebra $R\langle X\rangle^+$ without unit). It implies that there is a R-linear map $\mathit{red} \colon R\langle X_S\rangle^+ \to R\mathit{Irr}_S$, where $R\mathit{Irr}_S$ denotes the submodule of $R\langle X_S\rangle^+$ generated by irreducible monomials Irr_S, such that $w \equiv w'$ if, and only if, $\mathit{red}_S(w) = \mathit{red}_S(w')$. In particular, $\mathit{red}_S \circ \mathit{red}_S = \mathit{red}_S$. The element $\mathit{red}_S(p)$ is called the *normal form of* $p \in R\langle X_S\rangle^+$. The map $\overline{\mathit{red}}_S \colon R\wr S\wr \to R\mathit{Irr}_S$ defined by $\overline{\mathit{red}}_S(\pi(w)) = \mathit{red}_S$ is a linear isomorphism.

Moreover, Theorem 1.2 from [1] also states that $R\mathit{Irr}_S$ has a structure of an associative R-algebra given by $w \cdot w' = \mathit{red}_S(ww')$, and $R\wr S\wr \cong R\mathit{Irr}_S$ as R-algebras. In addition, let $w = x_1^{(i_1)} \cdots x_m^{(i_m)}, w' = y_1^{(j_1)} \cdots y_n^{(j_n)} \in \mathit{Irr}_S$. We have $\mathit{red}_S(ww') = ww'$ so that the operation "$\cdot$" is the usual concatenation of (non empty) words in the alphabet $\{\, x^{(i)} \colon x \in A \,\} = A \times \mathbb{N} \subseteq \mathit{Irr}_S$, and $R\wr S\wr$ is the free R-algebra $R\langle A \times \mathbb{N}\rangle^+$ generated by $A \times \mathbb{N}$. The corresponding derivation acts on $x^{(i)}$ as $\partial(x^{(i)}) = x^{(i+1)}$ for every $x \in A$, $i \geq 0$. Moreover, we have $R\wr A^*\wr \cong R\wr A^+\wr_1 \cong R\langle A \times \mathbb{N}\rangle_1^+ \cong R\langle A \times \mathbb{N}\rangle$. Using the abelianization functors, we find that $R\{\mathcal{S}(A)\} \cong \mathcal{Ab}(R\wr A^+\wr) \cong \mathcal{Ab}(R\langle A \times \mathbb{N}\rangle^+) \cong R[A \times \mathbb{N}]^+$, and $R\{\mathcal{M}(A)\} \cong \mathcal{Ab}(R\wr A^*\wr) \cong \mathcal{Ab}(R\langle A \times \mathbb{N}\rangle) \cong R[A \times \mathbb{N}]$. Therefore in this case we recover the usual differential commutative polynomials with variables in A.

5 The Universal Differential Envelope of an Algebra

In this section, again for the non-commutative as for the commutative cases, and unital as for non-unital cases, we present a way to form a universal differential envelope of an algebra, *i.e.*, to freely generate a differential algebra from a usual algebra. This is used to describe with precision the relation between the algebra of a semigroup (or monoid) and its differential algebra.

5.1 Non-commutative Case

We now prove that the forgetful functor from R-$\mathcal{DiffAlg}$ to R-$\mathcal{Alg}$ (respectively, from R-$\mathcal{DiffAlg}_1$ to R-$\mathcal{Alg}_1$) has a left adjoint. Let A be a R-algebra (respectively, a unital R-algebra). Let us consider the free R-algebra $R\langle A \times \mathbb{N}\rangle^+$ (respectively, $R\langle A \times \mathbb{N}\rangle$) generated by the alphabet $(A \times \mathbb{N})^+$ (respectively, $(A \times \mathbb{N})^*$). Let us consider the following relations on $R\langle A\times\mathbb{N}\rangle^+$ (respectively, $R\langle A\times\mathbb{N}\rangle$): for all $i\geq 0$, for all $x,y\in A$, and for all $\alpha\in R$, $(0_A)^{(i)}=0$, $(x+y)^{(i)}=x^{(i)}+y^{(i)}$, $(\alpha x)^{(i)}=\alpha(x)^{(i)}$ (in particular, $(-x)^{(i)}=-(x)^{(i)}$ so that $0_A^{(i)}=(x-x)^{(i)}=x^{(i)}+(-x)^{(i)}=x^{(i)}-(x^{(i)})=0$), $(x\times y)^{(i)}=\sum_{j=0}^{i}\binom{i}{j}x^{(j)}y^{(i-j)}$. In addition, $1_A^{(0)}=\epsilon$, and $1_A^{(i)}=0$ for every $i>0$, when A is unital.

Let $\mathcal{D}(A)$ be the quotient R-algebra $R\langle A\times\mathbb{N}\rangle^+/_I$ (respectively, $R\langle A\times\mathbb{N}\rangle/_I$) where I is the two-sided ideal generated by the above relations. We denote by $\equiv$ the corresponding congruence on $R\langle A\times\mathbb{N}\rangle^+$ (respectively, $R\langle A\times\mathbb{N}\rangle$), and $\pi\colon R\langle A\times\mathbb{N}\rangle^+\to\mathcal{D}(A)$ (respectively, $\pi\colon R\langle A\times\mathbb{N}\rangle\to\mathcal{D}(A)$) is the canonical epimorphism. According to Lemma 3, there exists a unique derivation $\partial\colon R\langle A\times\mathbb{N}\rangle^+\to R\langle A\times\mathbb{N}\rangle^+$ (respectively, $\partial\colon R\langle A\times\mathbb{N}\rangle\to R\langle A\times\mathbb{N}\rangle$) such that $\partial(x^{(i)})=x^{(i+1)}$ for every $i\geq 0$, $x\in A$. It is easy to check that I is actually a differential ideal, so that there is a unique derivation $\overline{\partial}$ on $\mathcal{D}(A)$ such that $\overline{\partial}\circ\pi(p)=\pi\circ\partial$.

Now, let (B,∂_B) be a differential R-algebra (respectively, unital differential R-algebra), and $\phi\colon A\to B$ be a homomorphism of R-algebras. Let $\phi_1\colon A\times\mathbb{N}\to B$ be the set-theoretic map defined by $\phi_1(x^{(i)})=\partial_B^i(\phi(x))$. Let $\phi_2\colon(A\times\mathbb{N})^+\to B$ (respectively, $\phi_2\colon(A\times\mathbb{N})^*\to B$) be defined as the unique semigroup (respectively, monoid) homomorphism extension of ϕ_1. Let $\phi_3\colon R\langle A\times\mathbb{N}\rangle^+\to B$ (respectively, $\phi_3\colon R\langle A\times\mathbb{N}\rangle\to B$) be the unique algebra homomorphism extension of ϕ_2. Let us check that ϕ_3 factors through the quotient algebra $\mathcal{D}(A)$. First of all it is easy to see that ϕ_3 commutes to the derivations, and that for every generating equation (u,v) of $\equiv$, $(\phi_3(u),\phi_3(v))$ is also a generating equation. Therefore there is a unique homomorphism of algebras $\widehat{\phi}\colon\mathcal{D}(A)\to B$ such that $\widehat{\phi}(\pi(p))=\phi_3(p)$ for every $p\in\mathcal{D}(A)$. Let us check that $\widehat{\phi}$ commutes with the derivations $\overline{\partial}$ and ∂_B. We have $\widehat{\phi}(\overline{\partial}(\pi(p)))=\widehat{\phi}(\pi(\partial(p)))=\phi_3(\partial(p))=\partial_B(\phi_3(p))=\partial_B(\widehat{\phi}(\pi(p)))$. Moreover it is easily checked that $\widehat{\phi}$ is unique. The map $q_A\colon A\to\mathcal{D}(A)$ defined by $q_A(x)=\pi(x^{(0)})$ is a homomorphism of R-algebras. The results above allow us to state the following lemma.

Lemma 15. *The differential R-algebra $(\mathcal{D}(A),\overline{\partial})$ is the free differential (respectively, free unital differential) R-algebra generated by the R-algebra (respectively, unital R-algebra) A,* i.e., *let (B,∂_B) be a differential R-algebra (respectively, a unital differential R-algebra), and $\phi\colon A\to B$ be a homomorphism of R-algebras. Then, there exists a unique homomorphism of differential R-algebras $\widehat{\phi}\colon(\mathcal{D}(A),\overline{\partial})\to(B,\partial_B)$ such that for every $x\in A$, $\widehat{\phi}(\pi(x^{(0)}))=\phi(x)$.*

The connection between the usual algebra of a semigroup and its differential algebra is stated in the following result.

Corollary 1. *Let S be a semigroup (respectively, monoid). Then, $(\mathcal{D}(R[S]),\overline{\partial})\cong(R\wr S\wr,\overline{\partial})$ as differential R-algebras (respectively, unital differential R-algebras).*

Proof. Let (B,∂_B) be a differential R-algebra (respectively, a unital differential R-algebra), and $\phi\colon S\to B$ be a homomorphism of semigroups (respectively, monoids). By the universal property of $R[S]$, there is a unique homomorphism of R-algebras $\widetilde{\phi}\colon R[S]\to B$ such that $\widetilde{\phi}(x)=\phi(x)$ for every $x\in S$. By Lemma 15, there is a unique homomorphism of differential R-algebras $\widehat{\widetilde{\phi}}\colon (\mathcal{D}(R[S]), \overline{\partial}) \to (B, \partial_B)$ such that for every $p \in R[S]$, $\widehat{\widetilde{\phi}}(\pi(p^{(0)})) = \widetilde{\phi}(p)$. Because $R[S]$ is free as a module with basis S, the last assertion is equivalent to the following: for every $x \in S$, $\widehat{\widetilde{\phi}}(\pi(x^{(0)})) = \widetilde{\phi}(x) = \phi(x)$. Indeed, let us assume that for every $x \in S$, $\widehat{\widetilde{\phi}}(\pi(x^{(0)})) = \widetilde{\phi}(x) = \phi(x)$. Let $p = \sum_{x \in S} \alpha_x x \in R[S]$. Then, $\widehat{\widetilde{\phi}}(\pi(p^{(0)})) = \widehat{\widetilde{\phi}}(\sum_{x \in S} \alpha_x \pi(x^{(0)})) = \sum_{x \in S} \alpha_x \widehat{\widetilde{\phi}}(\pi(x^{(0)})) = \sum_{x \in S} \alpha_x \phi(x) = \widetilde{\phi}(p)$. □

Remark 4. Using Corollary 1, we see that $\mathcal{D}(R\langle X\rangle^+) \cong R\wr X^+ \wr \cong R\langle X \times \mathbb{N}\rangle^+$ as algebras (see Subsection 4.7), for every set X. We also have $\mathcal{D}(R\langle X\rangle) \cong R\wr X^* \wr \cong R\langle X \times \mathbb{N}\rangle$.

Since $(A, 0)$ is a differential R-algebra (respectively, unital differential R-algebra), and $id_A\colon A \to A$ is an endomorphism of R-algebra, according to Lemma 15, there is a unique homomorphism of differential R-algebras, say $I\colon (\mathcal{D}(A), \overline{\partial}) \to (A, 0)$ such that $I \circ q_A = id_A$. Therefore, q_A is one-to-one, and A embeds into $\mathcal{D}(A)$ as a sub-algebra (respectively, unital sub-algebra).

Remark 5. The semigroup (respectively, monoid) algebra $R[S]$ embeds into $\mathcal{D}(R[S])$ which is isomorphic to $R\wr S \wr$ as a sub-algebra.

5.2 Commutative Case

It is also possible to define a similar construction in the commutative case. Let A be a commutative (respectively, commutative unital) R-algebra. We construct $(\mathcal{CD}(A), \overline{\partial})$ in a way similar to the previous construction $\mathcal{D}(A)$ in Subsection 5.1: first we consider the free commutative R-algebra $R[A \times \mathbb{N}]^+$ (respectively, $R[A \times \mathbb{N}]$) generated by $\mathcal{S}(A \times \mathbb{N})$ (respectively, $\mathcal{M}(A \times \mathbb{N})$). Then, we consider the same relations as in the previous construction, and we let $\mathcal{CD}(A)$ denote the quotient algebra $R[A \times \mathbb{N}]^+/_I$ (respectively, $R[A \times \mathbb{N}]/_I$) where I denotes the ideal generated by the relations. The corresponding congruence is again denoted by $\equiv$ and $\pi\colon R[A \times \mathbb{N}]^+ \to \mathcal{CD}(A)$ (respectively, $\pi\colon R[A \times \mathbb{N}] \to \mathcal{CD}(A)$) denotes the corresponding natural epimorphism. We know that there exists a unique derivation $\partial\colon R[A \times \mathbb{N}]^+ \to R[A \times \mathbb{N}]^+$ (respectively, $\partial\colon R[A \times \mathbb{N}] \to R[A \times \mathbb{N}]$) such that $\partial(x^{(i)}) = x^{(i+1)}$ for every $x \in A$, $i \geq 0$. Moreover I being a differential ideal, it follows that $\mathcal{CD}(A)$ admits a unique derivation $\overline{\partial}$ inherited from ∂. It is straightforward to obtain the following result by an easy variation of the proof of Lemma 15. Again the map $q_A\colon x \in A \mapsto \pi(x^{(0)}) \in \mathcal{CD}(A)$ is a homomorphism of algebras.

Lemma 16. *The commutative differential R-algebra $(\mathcal{CD}(A), \overline{\partial})$ is the free commutative (respectively, commutative unital) differential R-algebra generated by the commutative (respectively, commutative unital) R-algebra A,* i.e., *let (B, ∂_B) be a commutative (respectively, commutative unital) differential R-algebra, and $\phi\colon A \to B$ be a homomorphism of R-algebras. Then, there exists a unique homomorphism of commutative differential R-algebras $\widehat{\phi}\colon (\mathcal{CD}(A), \overline{\partial}) \to (B, \partial_B)$ such that for every $x \in A$, $\widehat{\phi}(\pi(x^{(0)})) = \phi(x)$.*

Corollary 2. *Let S be a commutative semigroup (respectively, monoid). Then, $(\mathcal{CD}(R[S]), \overline{\partial}) \cong (R\{S\}, \overline{\partial})$ as commutative differential R-algebras (respectively, commutative unital differential R-algebras).*

Proof. The proof is omitted since it is similar to that of Corollary 1. □

Remark 6. Using Corollary 2, we see that $\mathcal{CD}(R[X]^+) \cong R\{\mathcal{S}(X)\} \cong R[X \times \mathbb{N}]^+$ as algebras, for every set X. We also have $\mathcal{CD}(R[X]) \cong R\{\mathcal{M}(X)\} \cong R[X \times \mathbb{N}]$.

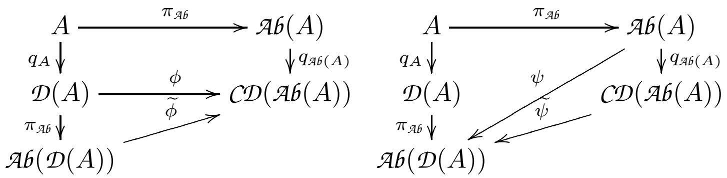
Since $(A, 0)$ is a commutative (respectively, commutative unital) differential R-algebra, and $id_A \colon A \to A$ is an endomorphism of commutative R-algebra, according to Lemma 16, there is a unique homomorphism of commutative differential R-algebras $I \colon (\mathcal{CD}(A), \overline{\partial}) \to (A, 0)$ such that $I \circ q_A = id_A$, so that q_A is one-to-one, and A embeds into $\mathcal{CD}(A)$ as a sub-algebra (respectively, unital sub-algebra).

Remark 7. The commutative semigroup (respectively, monoid) algebra $R[S]$ embeds into $\mathcal{CD}(R[S]) \cong R\{S\}$ as a sub-algebra (respectively, unital sub-algebra).

5.3 Links between Abelianization and the Free Differential Algebra

Lemma 17. *Let A be a R-algebra (unital or not). Then, $(\mathcal{Ab}(\mathcal{D}(A)), \partial_{\mathcal{Ab}(\mathcal{D}(A))}) \cong (\mathcal{CD}(\mathcal{Ab}(A)), \overline{\partial}_{\mathcal{Ab}(A)})$ as commutative R-algebras.*

Proof. The proof is due to the following commutative diagrams.



(10)

□

5.4 Links between Unitarization and the Free Differential Algebra

The following lemma is easily proved.

Lemma 18. *Let A be a non-unital R-algebra (respectively, non-unital commutative R-algebra). Then, as differential algebras, it holds that $(\mathcal{D}(A_1), \overline{\partial}) \cong (\mathcal{D}(A)_1, (\overline{\partial})_1)$ (respectively, as commutative differential algebras, it holds that $(\mathcal{CD}(A_1), \overline{\partial}) \cong (\mathcal{CD}(A)_1, (\overline{\partial})_1)$).*

6 General Differential Algebra

From now on, only differential algebra over a non-differential ring R has been explored. But it is also possible to take into account a derivation on the base ring, as for instance in example 2 the differential polynomials over a differential algebra (A, d_A). In this final section, we treat this kind of objects by the process of extension of scalars. This provides the final piece of the panorama of functorial constructions in the differential setting.

Let (A, d_A) be a usual unital and commutative R-differential algebra (recall that R is assumed to be both unital and commutative). Let B be a unital A-algebra. By restriction of scalars, it is also a unital R-algebra, and we assume that (B, ∂_B) is a unital differential R-algebra (*i.e.*, ∂_B is a R-linear derivation on B). Finally, we assume that for every $\alpha \in A$, $x \in B$, $\partial_B(\alpha x) = d_A(\alpha)x + \alpha\partial_B(x)$, while $\partial_B(\alpha x) = \alpha\partial_B(x)$ for every $\alpha \in R$, $x \in B$, since ∂_B is R-linear (we observe that it follows from $\partial_B(\alpha x) = d_A(\alpha)x + \alpha\partial_B(x)$ for α a scalar multiple of 1_A since d_A is R-linear and $d_A(1_A) = 0$). Such a structure (B, ∂_B) is a called a *unital differential R-algebra over* (A, d_A). If B is also commutative, then we obtain a *unital commutative differential R-algebra over* (A, d_A). As an example, (A, d_A) is a commutative unital differential R-algebra over itself by the Leibniz rule.

A homomorphism ϕ of unital (respectively, commutative unital) differential R-algebras over (A, d_A) from (B, ∂_B) to (C, ∂_C) is a homomorphism $\phi\colon B \to C$ for the underlying A-algebras such that $\phi \circ \partial_B = \partial_C \circ \phi$. If ϕ is such a homomorphism, then it is also a homomorphism of the underlying differential R-algebras. Unital (respectively, commutative unital) R-differential algebras over (A, d_A) and their homomorphisms form a category denoted by (A, d_A)-$\mathcal{DiffAlg}_1$ (respectively, (A, d_A)-$\mathcal{CDiffAlg}_1$).

There is an obvious forgetful functor from (A, d_A)-$\mathcal{DiffAlg}_1$ to R-$\mathcal{DiffAlg}_1$ (respectively, from (A, d_A)-$\mathcal{CDiffAlg}_1$ to R-$\mathcal{CDiffAlg}_1$). These functors are faithful and we will see that they have a left adjoint. It is a well-known fact that the categories R-$\mathcal{DiffAlg}_1$ and R-$\mathcal{CDiffAlg}_1$ admit a tensor product: let (B, ∂_B) and (C, ∂_C) be two unital (respectively, commutative unital) differential R-algebras. We define a R-linear derivation on the unital (respectively, commutative unital) R-algebras $B \otimes_R C$ as $\partial_B \otimes \partial_C\colon B \otimes_R C \to B \otimes_R C$ defined by $(\partial_B \otimes \partial_C)(b \otimes c) = \partial_B(b) \otimes c + b \otimes \partial_C(c)$. Let (B, ∂_B) be a usual differential (respectively, commutative differential) R-algebra. We provide to the unital (respectively, commutative unital) differential R-algebra $(A \otimes_R B, d_A \otimes \partial_B)$ a structure of unital (respectively, commutative unital) differential R-algebra over (A, d_A) as follows. The Abelian group $A \otimes_R B$ is given the trivial A-module structure by $a(a' \otimes b) = aa' \otimes b$. Because A is commutative, this A-module structure is compatible with the product of $A \otimes_R B$, this means that

$$\begin{aligned} a((a' \otimes b)(a'' \otimes b'')) &= a(a'a'' \otimes b'b'') \\ &= aa'a'' \otimes b'b'' \\ &= (a(a' \otimes b'))(a'' \otimes b'') \\ &= a'aa'' \otimes b'b'' \\ &= (a' \otimes b')(a(a'' \otimes b'')) \ . \end{aligned} \tag{11}$$

This A-algebra structure is commutative whenever B is. It is clear that by restriction of scalars, we recover the structure of R-module on $A \otimes_R B$. Finally, we have

$$\begin{aligned}(d_A \otimes \partial_B)(a(1_A \otimes 1_B)) &= (d_A \otimes \partial_B)(a \otimes 1_B) \\ &= d_A(a) \otimes 1_B + a \otimes \partial_B(1_B) \\ &= d_A(a) \otimes 1_B = d_A(a)(1_A \otimes 1_B)\ .\end{aligned} \tag{12}$$

Therefore, $(A \otimes_R B, d_A \otimes \partial_B)$ is a unital (respectively, commutative unital) differential R-algebra over (A, d_A). Let (C, ∂_C) be a unital differential R-algebra over (A, d_A). Let $\phi\colon (B, \partial_B) \to (C, \partial_C)$ be a homomorphism of differential R-algebras (C is assumed commutative if B is so). We define a R-linear map $\phi_1\colon A \times B \to C$ by $\phi_1(a, b) = a\phi(b)$ (well-defined since C is a A-algebra). Therefore, $\phi_2\colon A \otimes_R B \to C$ given by $\phi_2(a \otimes b) = \phi_1(a, b)$ is R-linear.

Moreover it is easily checked that it commutes with the derivations, and is unique with those properties. It is also clear that the map $i_B\colon b \in B \mapsto 1_A \otimes b \in A \otimes_R B$ is a homomorphism of differential algebras from (B, ∂_B) to $(A \otimes_R B, d_A \otimes \partial_B)$. This means that the following result holds.

Lemma 19. *Let (B, ∂_B) be a unital (respectively, commutative unital) differential R-algebra and (A, d_A) be a commutative differential R-algebra. Then, for every unital (respectively, commutative unital) differential R-algebra (C, ∂_C) over (A, d_A) and every homomorphism $\phi\colon (B, \partial_B) \to (C, \partial_C)$ of differential R-algebras, there exists a unique homomorphism of differential R-algebras $\phi_2\colon (A \otimes B, d_A \otimes \partial_B) \to (C, \partial_C)$ such that $\phi_2(1_A \otimes b) = \phi(b)$ for every $b \in B$.*

Let (B, ∂_B) be a unital differential R-algebra over (A, d_A). Let $\mathcal{Ab}(B)$ be the abelianization of B as a unital A-algebra. Let $\pi_{\mathcal{Ab}}\colon B \to \mathcal{Ab}(B)$ be the natural epimorphism (it is a homomorphism of unital A-algebras). It is easy to see that the R-linear map (by restriction of scalars from A to R) $\pi_{\mathcal{Ab}} \circ \partial_B\colon B \to \mathcal{Ab}(B)$ passes to the quotient. Therefore, there is a unique R-derivation $\partial_{\mathcal{Ab}(B)}$ on $\mathcal{Ab}(B)$ such that $\partial_{\mathcal{Ab}(B)} \circ \pi_{\mathcal{Ab}} = \pi_{\mathcal{Ab}} \circ \partial_B$.

Moreover, $(\mathcal{Ab}(B), \partial_{\mathcal{Ab}(B)})$ becomes a commutative unital differential algebra over (A, d_A). Then, it follows that $(\mathcal{Ab}(B), \partial_{\mathcal{Ab}(B)})$ is the abelianization of (B, ∂_B). The following two results are easily proved.

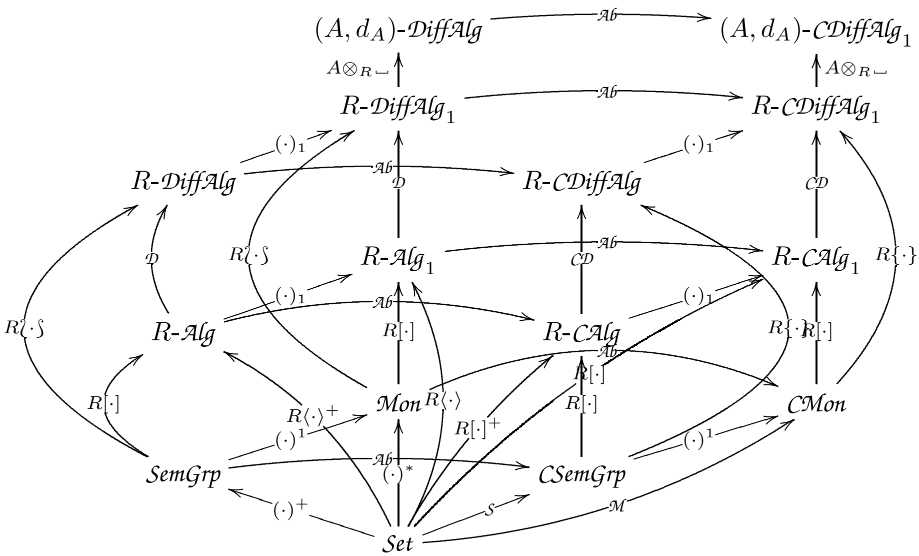
Lemma 20. *Let (B, ∂_B) be a unital differential R-algebra, and (A, d_A) be a commutative unital differential R-algebra. Then, $(A \otimes_R \mathcal{Ab}(B), d_A \otimes \partial_{\mathcal{Ab}(B)})$ is the abelianization of $(A \otimes_R B, d_A \otimes \partial_B)$ (where $(\mathcal{Ab}(B), \partial_{\mathcal{Ab}(B)})$ is the abelianization of (B, ∂_B) as a unital differential R-algebra).*

Lemma 21. *Let M is any monoid (respectively, commutative monoid). Then, $A \otimes_R R\wr M \wr \cong A\wr M \wr$ (respectively, $A \otimes_R R\{M\} \cong A\{M\}$) as unital A-algebras (where we equip $A \otimes_R R\wr M \wr$, respectively, $A \otimes_R R\{M\}$, with its A-algebra structure). Moreover, let us define the R-linear derivation ∂' on $A\wr M \wr$ (respectively, $A\{M\}$) by $\partial'(a\pi(x^{(i)})) = d_A(a)\pi(x^{(i)}) + a\pi(x^{(i+1)})$. Then, $(A \otimes_R R\wr M \wr, d_A \otimes \overline{\partial}) \cong (A\wr M \wr, \partial')$ (respectively, $(A \otimes_R R\{M\} d_A \otimes \overline{\partial}) \cong (A\{M\}, \partial')$) as unital differential R-algebras over (A, d_A).*

Using the results from Subsection 4.7, for any commutative unital differential R-algebra (A, d_A), and for any set X, $A\wr X^* \wr \cong A\langle X \times \mathbb{N} \rangle$ (respectively,

$A\{\mathcal{M}(X)\} \cong A[X \times \mathbb{N}])$ as unital A-algebras. Moreover, under this isomorphism, in the commutative case, we recover the usual differential polynomials over (A, d_A).

The previous Lemmas 20 and 21, together with all the previous results of this paper, allow us to give a complete view on our panorama as a commutative diagram of functors (every path from a given source to a given target we can follow in this large graph gives the same functor).



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