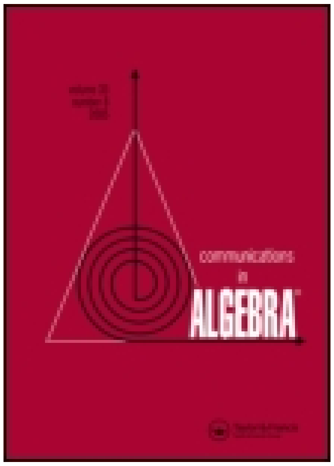

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Free Monoids over Semigroups in a Monoidal Category: Construction and Applications

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FREE MONOIDS OVER SEMIGROUPS IN A MONOIDAL CATEGORY: CONSTRUCTION AND APPLICATIONS

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The adjunction of a unit to an algebraic structure with a given binary associative operation is discussed by interpreting such structures as semigroups and monoids respectively in a monoidal category. This approach then allows for results on the adjunction of counits to coalgebraic structures with a binary co-associative co-operation as well. Special attention is paid to situations where a given coalgebraic structure induces a "dual" algebraic one; here the compatibility of adjoining (co)units and dualization is examined. The extension of this process to starred algebraic structures and to monoid actions is discussed as well. Particular emphasis is given to examples from many areas of mathematics.

Key Words: (Banach) algebra; Coring; Monoid; Monoidal category; Ring.

2010 Mathematics Subject Classification: 16T15, 18D10.

1. INTRODUCTION

It is well known that one can adjoin a unit to a semigroup and other algebraic structures with an associative binary operation. The constructions are often easy and all similar. In fact, the existence of these constructions is, in some of the best known cases as, e.g., for monoids, rings and algebras, and for the internal versions of these in a locally presentable category as well (see [34]), a special instance of the fact that algebraic functors have left adjoints. But a construction according to this idea would be unnecessarily complicated. Since the concepts of semigroups and monoids are most naturally discussed in the realm of monoidal categories, we discuss in this note the adjunction of units within this theory, where it turns out to be a rather straightforward generalization of the familiar constructions.

Examples of this *unitarization* process then include—amongst others—the adjunction of a unit to algebras, bialgebras, R-rings, semirings, Banach algebras, and graded algebras.

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Since the chosen approach allows for categorical dualization, we also provide examples of *counitarization*, i.e., the adjunction of a counit, as for example in the cases of coalgebras and R-corings.

The familiar construction of the dual algebra of a coalgebra is a paradigmatic example of situations, where one can assign to a given coalgebraic structure a dual algebraic one. We therefore investigate the problem to what extent the constructions of dualization and (co)unitarization commute.

Moreover, it will be shown how to extend the unitarization process to starred algebraic structures like rings with involution and to semigroup actions and monoid actions.

Though not all of our results may be very deep from a categorical point of view (in particular Theorem 5), we are confident about the interest of their implications and applications in quite a number of areas of mathematics. In support of this conviction, we add a long list of examples.

2. NOTATION AND PREREQUISITES

Throughout, $\mathbb{C} = (\mathbf{C}, -\otimes-, I, \alpha, \lambda, \varrho)$ denotes a monoidal category, where $-\otimes-$ denotes a bifunctor on the category $\mathbf{C}$, I a $\mathbf{C}$-object, and the natural isomorphisms $(A\otimes B)\otimes C \xrightarrow{\alpha_{A,B,C}} A\otimes(B\otimes C)$, $C\otimes I \xrightarrow{\rho_C} C$, and $I\otimes \xrightarrow{\lambda_C} C$ the associativity and right and left unit constraints, respectively. If $\mathbb{C}$ even is symmetric monoidal, the symmetry will be denoted by $\sigma = (C\otimes D \xrightarrow{\sigma_{CD}} D\otimes C)_{C,D}$. $\mathbb{C}$ is called *cartesian* if the tensor product is given by categorical product and I is the terminal object of $\mathbf{C}$.

A *monoid in* $\mathbb{C}$ is a triple $(C, C\otimes C \xrightarrow{m} C, I \xrightarrow{e} C)$ such that the diagrams

$$\begin{array}{ccc} C\otimes C\otimes C & \xrightarrow{m\otimes 1_C} & C\otimes C \\ {\scriptstyle 1_C\otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ C\otimes C & \xrightarrow[m]{} & C \end{array} \qquad \begin{array}{ccccc} C\otimes I & \xrightarrow{1_C\otimes e} & C\otimes C & \xleftarrow{e\otimes 1_C} & I\otimes C \\ & {\scriptstyle \rho_C}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle \lambda_C} & \\ & & C & & \end{array}$$

commute, while a *monoid morphism* $(C, m, e) \longrightarrow (C', m', e')$ is any $f : C \to C'$ making the diagrams

$$\begin{array}{ccc} C\otimes C & \xrightarrow{m} & C \\ {\scriptstyle f\otimes f}\downarrow & & \downarrow{\scriptstyle f} \\ C'\otimes C' & \xrightarrow[m']{} & C' \end{array} \qquad \begin{array}{ccc} I & \xrightarrow{e} & C \\ & {\scriptstyle e'}\searrow & \downarrow{\scriptstyle f} \\ & & C' \end{array}$$

commutative. This defines the category $\mathbf{Mon}\mathbb{C}$ of monoids in $\mathbb{C}$.

The category $\mathbf{Comon}\mathbb{C}$ of comonoids over $\mathbb{C}$ is defined to be $(\mathbf{Mon}\mathbb{C}^{\mathsf{op}})^{\mathsf{op}}$, the dual of the category of monoids in $\mathbb{C}^{\mathsf{op}}$. [1] *Comonoids* thus are triples $(C, C \xrightarrow{\mu} C\otimes$

[1] Here $\mathbb{C}^{\mathsf{op}}$ denotes the monoidal category $(\mathbf{C}^{\mathsf{op}}, -\otimes-, I)$, i.e., the dual of $\mathbf{C}$ with the given tensor product.

$C,\ C \stackrel{\epsilon}{\longrightarrow} I$) such that the diagrams

$$\begin{CD} C @>{\mu}>> C\otimes C \\ @V{\mu}VV @VV{1_C\otimes\mu}V \\ C\otimes C @>>{\mu\otimes 1_C}> C\otimes C\otimes C \end{CD} \qquad \begin{array}{ccccc} C\otimes I & \xleftarrow{1_C\otimes\epsilon} & C\otimes C & \xrightarrow{\epsilon\otimes 1_C} & I\otimes C \\ & \searrow^{\rho_C^{-1}} & \uparrow{\scriptstyle\mu} & \nearrow_{\lambda_C^{-1}} & \\ & & C & & \end{array}$$

commute, while a *comonoid morphism* $(C,\mu,\epsilon) \longrightarrow (C',\mu',\epsilon')$ is any $f : C \to C'$ making the diagrams

$$\begin{CD} C @>{\mu}>> C\otimes C \\ @V{f}VV @VV{f\otimes f}V \\ C' @>>{\mu'}> C'\otimes C' \end{CD} \qquad \begin{array}{ccc} C & \xrightarrow{\epsilon} & I \\ {\scriptstyle f}\downarrow & \nearrow_{\epsilon'} & \\ C' & & \end{array}$$

commute.

A monoid (C, m, e) is called *commutative* iff $m = m \circ \sigma_{C,C}$ with $\sigma_{C,C} : C \otimes C \to C \otimes C$ the symmetry; dually, a comonoid (C, μ, ϵ) is called *cocommutative*, provided that $\mu = \sigma_{C,C} \circ \mu$. By ${}_c\mathbf{Mon}\mathbb{C}$ and ${}_{coc}\mathbf{Comon}\mathbb{C}$, we denote the categories of commutative monoids and cocommutative comonoids, respectively, with all (co)monoid morphisms as morphisms. One has ${}_{coc}\mathbf{Comon}\mathbb{C} = ({}_c\mathbf{Mon}\mathbb{C}^{\mathsf{op}})^{\mathsf{op}}$.

Omitting the unit e of a monoid as well as the respective axioms one obtains the categories $\mathbf{Sgr}\mathbb{C}$ and ${}_c\mathbf{Sgr}\mathbb{C}$ of (commutative) semigroups (S, s) in $\mathbb{C}$. Dually, one has the categories $\mathbf{Cosgr}\mathbb{C}$ and ${}_{coc}\mathbf{Cosgr}\mathbb{C}$ of (co-commutative) co-semigroups.

Remark 1. Note that if would have been more correct to write the axioms describing associativity of a monoid (and, dually, the co-associativity axiom for a comonoid) as

$$\begin{array}{ccccc} & & (C\otimes C)\otimes C & \xrightarrow{m\otimes 1_C} & C\otimes C \\ & \nearrow^{\alpha_{C,C,C}} & & & \\ (C\otimes C)\otimes C & & & & \downarrow{\scriptstyle m} \\ {\scriptstyle 1_C\otimes m}\downarrow & & & & \\ C\otimes C & & \xrightarrow[m]{\qquad\qquad} & & C \end{array}.$$

This is legitimized by Mac Lane's coherence theorem [27]. For details, see, e.g., [3, 1.4].

The following situations are well-known examples of these concepts. Further examples are discussed in Section 4.

1. If $\mathbb{C}$ is the (cartesian) monoidal category Set of sets and maps, then $\mathbf{Mon}\mathbb{C}$ is the category of ordinary monoids, while $\mathbf{Sgr}\mathbb{C}$ is the category of ordinary semigroups.
2. If $\mathbb{C}$ is the monoidal category $\mathbf{Mod}_R$ of R-modules and R-linear maps for a commutative unital ring R with its usual tensor product, then $\mathbf{Mon}\mathbb{C}$ is the

category ${}_1\mathrm{Alg}_R$ of unital R-algebras and $\mathbf{Comon}\mathbb{C}$ is the category ${}_\epsilon\mathbf{Coalg}_R$ of counital R-coalgebras.

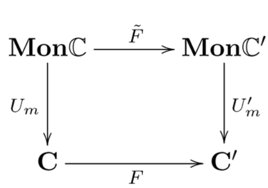
We briefly recall the following definitions and facts, too, which are fundamental for this note.

Definition 2. Let $\mathbb{C} = (\mathbf{C}, -\otimes -, I)$ and $\mathbb{C}' = (\mathbf{C}', -\otimes' -, I')$ be monoidal categories. A *lax (weak) monoidal functor from* $\mathbb{C}$ *to* $\mathbb{C}'$ is a triple (F, Φ, ϕ), where $F : \mathbf{C} \to \mathbf{C}'$ is a functor, $\Phi_{C_1,C_2} : FC_1 \otimes' FC_2 \to F(C_1 \otimes C_2)$ is a natural transformation and $\phi : I' \to FI$ is a $\mathbf{C}$-morphism, subject to certain coherence conditions (see e.g. [40]). A lax monoidal functor is called *strong monoidal* (resp., *strict monoidal*), if Φ and ϕ are isomorphisms (resp., identities).

Proposition 3. *Let* $(F, \Phi, \phi) : \mathbb{C} \to \mathbb{C}'$ *be a lax monoidal functor.*

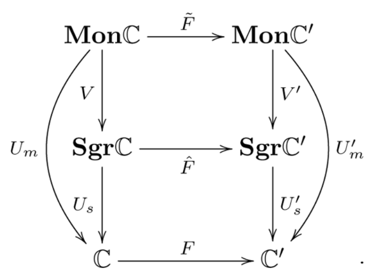
$$\tilde{F}(M, m, e) = (FM, FM \otimes FM \xrightarrow{\Phi_{M,M}} F(M \otimes M) \xrightarrow{Fm} FM, I' \xrightarrow{\phi} FI \xrightarrow{Fe} FM),$$

and $\tilde{F}f = Ff$ *define an induced functor* $\tilde{F} : \mathbf{Mon}\mathbb{C} \to \mathbf{Mon}\mathbb{C}'$, *such that the diagram*



commutes (*with forgetful functors* U_m *and* U'_m).

The same obviously holds for the categories of semigroups over $\mathbb{C}$ and $\mathbb{C}'$, respectively. Write then $\hat{F}$ instead of $\tilde{F}$. We thus have the following commutative diagram (with forgetful functors V, V', U_s, and U'_s):



Remark 4. By duality a strong monoidal functor F induces functors $F_m : \mathbf{Comon}\mathbb{C} \to \mathbf{Comon}\mathbb{C}'$ and $F_s : \mathbf{Cosgr}\mathbb{C} \to \mathbf{Cosgr}\mathbb{C}'$ commuting with the forgetful functors as in the diagram above.

3. ADJOINING A UNIT TO A SEMIGROUP

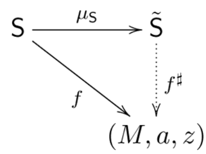
Theorem 5. *Let* $\mathbb{C} = (\mathbf{C}, -\otimes -, I)$ *be a monoidal category where the following statemenst hold*:

1. $\mathbf{C}$ *has binary coproducts*;
2. *All functors* $C \otimes -$ *and* $- \otimes C$ *preserve coproducts of the form* $S + I$.

Then the forgetful functor $\mathbf{Mon}\mathbb{C} \to \mathbf{Sgr}\mathbb{C}$ *has a left adjoint, called the unitarization functor.*

If $\mathbb{C}$ *is a symmetric monoidal category the following holds, in addition: If the semigroup* (S, m) *is commutative so is its unitarization* $(\tilde{S}, \tilde{m}, \tilde{e})$.

In more detail, for every semigroup $\mathsf{S} = (S, m)$ there exists a monoid $\tilde{\mathsf{S}} = (\tilde{S}, \tilde{m}, \tilde{e})$ and a morphism of semigroups $\mu_{\mathsf{S}} : \mathsf{S} \to \tilde{\mathsf{S}}$, such that every morphism of semigroups $f : \mathsf{S} \to (M, a, z)$ from S to (the underlying semigroup of) a monoid admits a unique monoid morphism $f^\sharp : \tilde{\mathsf{S}} \to (M, a, z)$ making the following diagram commute:



Proof. Put $\tilde{S} := S + I$, the coproduct of S and I, with coproduct injections

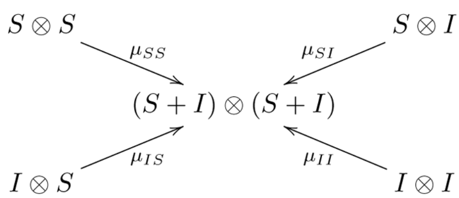
$$S \xrightarrow{\mu_S} S + I \xleftarrow{\mu_I} I.$$

$S + I$ can be equipped with $\mathbf{C}$-morphims $\tilde{m} : \tilde{S} \otimes \tilde{S} \to \tilde{S}$ and $\tilde{e} : I \to \tilde{S}$ such that $\tilde{\mathsf{S}} = (\tilde{S}, \tilde{m}, \tilde{e})$ is a monoid in $\mathbb{C}$ and $\mu_S : \mathsf{S} \to \tilde{\mathsf{S}}$ is the required morphism of semigroups as follows.

The unit $\tilde{e} : I \to \tilde{S}$ is defined as

$$\tilde{e} = I \xrightarrow{\mu_I} S + I$$

In order to define the multiplication $\tilde{m} : \tilde{S} \otimes \tilde{S} \to \tilde{S}$, observe first that, by assumption 7, $(S + I) \otimes (S + I)$ has a coproduct presentation



with

$$\mu_{SS} = \mu_S \otimes \mu_S \quad \mu_{SI} = \mu_S \otimes \mu_I \quad \mu_{IS} = \mu_I \otimes \mu_S \quad \mu_{II} = \mu_I \otimes \mu_I.$$

If we put

$$m_{SS} = S \otimes S \xrightarrow{m} S \qquad m_{SI} = S \otimes I \xrightarrow{\rho_S} S \qquad m_{IS} = I \otimes S \xrightarrow{\lambda_S} S \qquad m_{II} = I \otimes I \xrightarrow{\lambda_I} I \tag{1}$$

the multiplication $\tilde{m} : (S+I) \otimes (S+I) \to (S+I)$, then is defined component-wise as follows:

$$\tilde{m} \circ \mu_{JK} = \mu_S \circ m_{JK} \quad \text{and} \quad \tilde{m} \circ \mu_{II} = \mu_I \circ m_{II} \tag{2}$$

for $(I, I) \neq (J, K) \in \{I, S\}^2$.

$(S+I, \tilde{m}, \tilde{e})$ is a monoid in $\mathbb{C}$, provided that the following diagrams commute:

$$\begin{array}{ccccc} I\otimes(S+I) & \xrightarrow{\tilde e\otimes(S+I)} & (S+I)\otimes(S+I) & \xleftarrow{(S+I)\otimes\tilde e} & (S+I)\otimes I \\ & \searrow^{\lambda_{S+I}} & \big\downarrow{\scriptstyle\tilde m} & \swarrow_{\rho_{S+I}} & \\ & & S+I & & \end{array} \tag{3}$$

$$\begin{array}{ccc} (S+I)\otimes(S+I)\otimes(S+I) & \xrightarrow{(S+I)\otimes\tilde m} & (S+I)\otimes(S+I) \\ {\scriptstyle\tilde m\otimes(S+I)}\big\downarrow & & \big\downarrow{\scriptstyle\tilde m} \\ (S+I)\otimes(S+I) & \xrightarrow[\tilde m]{} & S+I\ . \end{array} \tag{4}$$

Concerning Diagram (3), we use the coproduct presentation of $I \otimes (S+I)$

$$I\otimes S \xrightarrow{I\otimes\mu_S} I\otimes(S+I) \xleftarrow{I\otimes\mu_I} I\otimes I\ .$$

Commutativity of the left-hand triangle of Diagram (3) then is equivalent to the equations

$$\tilde{m} \circ (\tilde{e} \otimes (S+I)) \circ (I \otimes \mu_S) = \lambda_{S+I} \circ (I \otimes \mu_S), \tag{5}$$

$$\tilde{m} \circ (\tilde{e} \otimes (S+I)) \circ (I \otimes \mu_I) = \lambda_{S+I} \circ (I \otimes \mu_I). \tag{6}$$

Since $\tilde{e} = \mu_I$ by definition, this can be read off the following diagrams with $J = I$ or $J = S$, where the outer frame commutes by definition of $\tilde{m}$ and the left-hand square commutes, since λ is a natural transformation:

$$\begin{array}{ccccc} & & \overset{\mu_{JI}}{\frown} & & \\ I\otimes J & \xrightarrow{I\otimes\mu_J} & I\otimes(S+I) & \xrightarrow{\mu_I\otimes(S+I)} & (S+I)\otimes(S+I) \\ & \searrow^{\mu_{IJ}=\lambda_J} & & \searrow^{\lambda_{S+I}} & \big\downarrow{\scriptstyle\tilde m} \\ & & S & \xrightarrow[\mu_J]{} & S+I\ . \end{array}$$

Similarly for the right-hand diagram. Thus, $\tilde{e}$ is a unit with respect to $\tilde{m}$.

Concerning Diagram (4), we use the coproduct presentation

$$\mu_{JKL} := J \otimes K \otimes L \stackrel{\mu_J \otimes \mu_K \otimes \mu_L}{\longrightarrow} \otimes^3(S+I) \quad J, K, L \in \{S, I\}$$

of $\otimes^3(S+I) := (S+I) \otimes (S+I) \otimes (S+I)$. Commutativity of Diagram (3) then is equivalent to the equations

$$\tilde{m} \circ ((S+I) \otimes \tilde{m}) \circ \mu_{JKL} = \tilde{m} \circ (\tilde{m} \otimes (S+I)) \circ \mu_{JKL} \quad \text{for all} \quad J, K, L \in \{S, I\}.$$

By functoriality of $- \otimes -$ and Eq. (2), one gets

$$((S+I) \otimes \tilde{m}) \circ \mu_{JKL} = \mu_{JL \vee K} \circ (J \otimes m_{KL}),$$

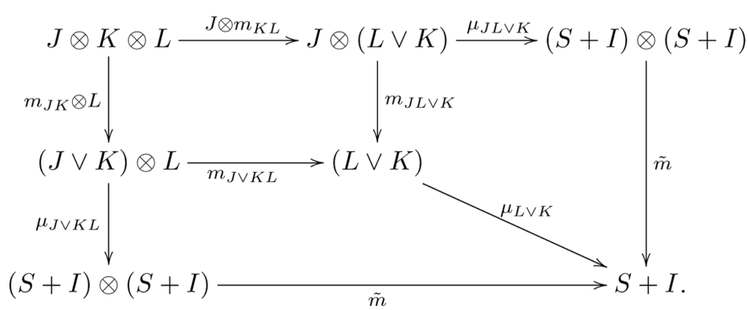
where $L \vee K$ denotes the target of m_{KL}. Thus,

$$((S+I) \otimes \tilde{m}) \circ \mu_{JKL} = \mu_{JL \vee K} \otimes (J \otimes m_{KL}).$$

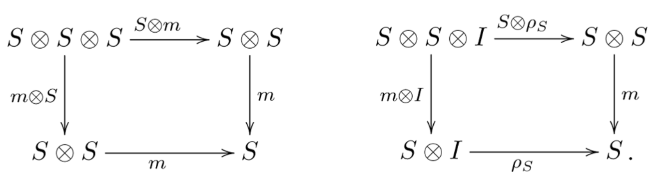
Similarly,

$$(\tilde{m} \otimes (S+I)) \circ \mu_{JKL} = \mu_{J \vee KL} \circ (m_{JK} \otimes L).$$

We thus need to check commutativity of the outer frames of the following diagrams, for all $(J, K, L) \in \{S, I\}$, where we only need to specify the various maps m_{JK} according to Eq. (1):



The right-hand and lower cells commute by definition of $\tilde{m}$, such that we only have to care about the top left squares. These are as follows:



Commutativity in the case $JKL = SSS$ is simply associativity of m, while commutativity in the case SSI follows from naturally of ρ, which gives $m \circ \rho_{S \otimes S} = \rho_S \circ (m \otimes I)$, and coherence (see, e.g., [27]), which here gives $\rho_{S \otimes S} = S \otimes \rho_S$. The remaining cases are similar. Thus, $\tilde{\mathsf{S}} = (\tilde{S}, \tilde{m}, \tilde{e})$ is a monoid in $\mathbb{C}$.

That $\mu_{\mathsf{S}} := \mu_S : \mathsf{S} \to \tilde{\mathsf{S}}$ is a morphism of semigroups is equivalent to Eq. (2) for $JK = SS$, since $\mu_{SS} = \mu_S \otimes \mu_S$.

Let now $\mathsf{M} = (M, M \otimes M \xrightarrow{a} M, I \xrightarrow{z} M)$ be a monoid in $\mathbb{C}$ and $f : (S, m) \to (M, a)$ a morphism of semigroups. Denote by $f^\sharp : (S + I) \to M$ the $\mathbb{C}$-morphism with components f and z, i.e., the unique morphism making the diagram

$$\begin{array}{ccccc} & & S+I & & \\ & \overset{\mu_S}{\nearrow} & \Big\downarrow{\scriptstyle f^\sharp} & \overset{\mu_I}{\nwarrow} & \\ S & & & & I \\ & \underset{f}{\searrow} & & \underset{z}{\swarrow} & \\ & & M & & \end{array} \tag{7}$$

commute. $f^\sharp$ preserves the unit by its very definition, since μ_I is the unit of S. Thus, the definition of $f^\sharp$ immediately shows that it is the unique monoid morphism factoring through μ_S, provided that it preserves the multiplication, that is, that the following diagram commutes:

$$\begin{array}{ccc} (S+I)\otimes(S+I) & \xrightarrow{f^\sharp\otimes f^\sharp} & M\otimes M \\ {\scriptstyle \tilde{m}}\Big\downarrow & & \Big\downarrow{\scriptstyle a} \\ S+I & \xrightarrow[f^\sharp]{\qquad} & M\,. \end{array}$$

By means of the coproduct presentation of $(S + I) \otimes (S + I)$ as above, this follows in a straightforward fashion, using—besides the fact that f is a semigroup morphism—functoriality of $- \otimes -$, naturality of ρ and λ, and, Eqs. (2) and (7).

Concerning the final statement, one observes first that the symmetry σ on $(S + I) \otimes (S + I)$ is the coproduct of the symmetries $\sigma_{SS}, \sigma_{SI}, \sigma_{IS}, \sigma_{II}$. Using Eqs. (1) and (2), one now gets $\tilde{m} \circ \sigma = \tilde{m}$ as required. □

Remark 6.

1. The hypothesis of the above theorem is satisfied if the category $\mathbb{C}$ is monoidal closed, since then the tensor functors, being left adjoints, preserve (all) coproducts. This is the case in particular if **C** is the category of algebras for a commutative algebraic theory, since such categories are symmetric monoidal closed by a result of Linton [25] (see also [9]). Examples of this kind are **Ab**, ${}_R\mathbf{Mod}_R$, the category of abelian groups and, more generally, $\mathbf{Mod}_R$, the category of modules over a commutative unital ring R, but also ${}_c\mathbf{Mon}$, the category of commutative monoids and the category Set of sets with binary product as tensor product (see Examples 4.1.1, 4.3.1, 4.3.2, 4.3.3). Other monoidal closed categories (not in this class) we are going to use are the category $\mathbf{Ban}_1$ of Banach spaces and short maps (with the projective tensor product), the category ${}_R\mathbf{Mod}_R$ of bimodules over a noncommutative unital ring R (here, due to the lack of symmetry, one more precisely should say "left closed" and "right closed"), and **Hilb**, the category of (real or complex) Hilbert spaces, which—though closed only

in the finite dimensional case—also satisfies the assumptions of the theorem (see [37]).

2. The following example shows that preservation of arbitrary (binary) coproducts is not required (as is obvious from the proof above). Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be an arbitrary bounded lattice. Considered as a category, $\mathbf{L}$ is (cartesian) monoidal with binary products given by $\wedge$, terminal object 1, and binary coproducts given by $\vee$. As is easily seen, this category satisfies Assumption 2 of Theorem 5, though not the preservation of all binary coproducts by the tensor functors $x \wedge -$ and $- \wedge x$, $x \in L$ (this would require the lattice to be distributive). In fact, every element x in L is a semigroup in $\mathbf{L}$, the top element 1 is the only monoid, and the morphism corresponding to $x \leq 1$ is the monoid reflection of x (as also follows from Theorem 5).

Dually to Theorem 5, one has the following theorem.

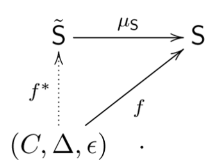
Theorem 7. *Let $\mathbb{C}$ be a monoidal category where the following statements hold*:

1. $\mathbf{C}$ *has binary products*;
2. *All functors $C \otimes -$ and $- \otimes C$ preserve products of the form $S \times I$.*

Then the forgetful functor $\mathbf{Comon}\mathbb{C} \to \mathbf{Cosgr}\mathbb{C}$ *has a right adjoint, called the co-unitarization functor.*

If $\mathbb{C}$ is a symmetric monoidal category, the following statements holds, in addition: *If the cosemigroup (S, δ) is cocommutative, so is its counitarization $(\tilde{S}, \tilde{\delta}, \tilde{\epsilon})$.*

In more detail, for every cosemigroup $\mathsf{S} = (S, \delta)$ there exists a comonoid $\tilde{\mathsf{S}} = (\tilde{S}, \tilde{\delta}, \tilde{\epsilon})$ and a morphism of cosemigroups $\mu_{\mathsf{S}} : \tilde{\mathsf{S}} \to \mathsf{S}$, such that every morphism of cosemigroups $f : (C, \Delta, \epsilon) \to \mathsf{S}$ from (the underlying cosemigroup of) a comonoid (C, Δ, ϵ) to S admits a unique comonoid morphism $f^* : (C, \Delta, \epsilon) \to \tilde{\mathsf{S}}$ making the following diagram commute:



Remark 8. The requirement in the theorem above, that tensoring preserves binary products of the form $S \times I$, may seem restrictive. Note, however, that a stronger condition (namely, that the functors $C \otimes -$ and $- \otimes C$ preserve all binary products) is satisfied in every additive category with biproducts, where the functors $C \otimes -$ and $- \otimes C$ are additive, thus, for example in $\mathbf{Mod}_R$ and in ${}_R\mathbf{Mod}_R$. The following simple lemma, that extends a result of [6], shows that the categories ${}_1\mathbf{Alg}_R$ and ${}_{c,1}\mathbf{Alg}_R$ share this property as well.

Lemma 9. *In the categories ${}_1\mathbf{Alg}_R$ and ${}_{c,1}\mathbf{Alg}_R$, the functors $\mathsf{A} \otimes -$ preserve binary products.*

Proof. Because ${}_{c,1}\mathbf{Alg}_R$, the category of commutative unital algebras, is closed in ${}_1\mathbf{Alg}_R$ under tensoring and products, it suffices to consider ${}_1\mathbf{Alg}_R$. Since the forgetful functor $|-| : {}_1\mathbf{Alg}_R \to \mathbf{Mod}_R$ creates products, it suffices to show that, whenever

$$\mathsf{B} \xleftarrow{p_{\mathsf{B}}} \mathsf{B} \times \mathsf{C} \xrightarrow{p_{\mathsf{C}}} \mathsf{C} \tag{8}$$

is a product in ${}_1\mathbf{Alg}_R$, then

$$|\mathsf{A} \otimes \mathsf{B}| \xleftarrow{|\mathsf{A}\otimes p_{\mathsf{B}}|} |\mathsf{A} \otimes (\mathsf{B} \times \mathsf{C})| \xrightarrow{|\mathsf{A}\otimes p_{\mathsf{C}}|} |\mathsf{A} \otimes \mathsf{C}| \tag{9}$$

is a product in $\mathbf{Mod}_R$.

But this follows easily since the functor $|-|$ is strict monoidal and products in $\mathbf{Mod}_R$ are biproducts and, thus, preserved by tensoring. □

Theorem 5 is compatible with the functors introduced in Proposition 3 in the following sense.

Theorem 10. *Let $\mathbb{C}$ and $\mathbb{C}'$ be monoidal categories with binary coproducts, where coproducts of the form $S + I$ are preserved by all functors $C \otimes -$ and $- \otimes C$. If $(F, \Phi, \phi) : \mathbb{C} \to \mathbb{C}'$ is a lax monoidal functor, such that F preserves binary coproducts and ϕ is an isomorphism, then the diagram*

$$\begin{array}{ccc} \mathbf{Mon}\mathbb{C} & \xrightarrow{\tilde{F}} & \mathbf{Mon}\mathbb{C}' \\ {\scriptstyle A}\uparrow & & \uparrow{\scriptstyle A'} \\ \mathbf{Sgr}\mathbb{C} & \xrightarrow[\hat{F}]{} & \mathbf{Sgr}\mathbb{C}' \end{array}$$

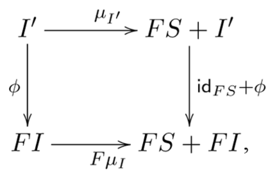
commutes (up to a natural isomorphism), where A and A' denote the left adjoints of V and V' respectively.

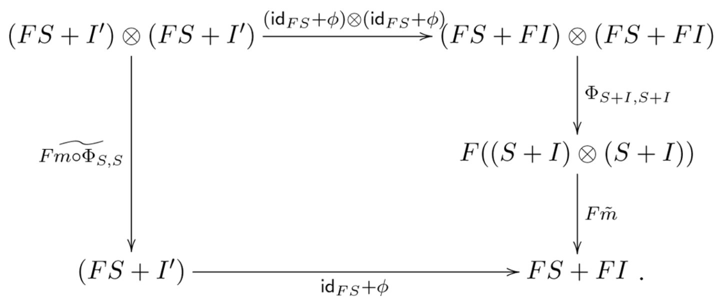
Some simple illustrations of the situation described here can be found in Example 4.1.2, while more interesting applications will be given given in Example 4.3.

Proof. One has, for a semigroup (S, m) in $\mathbb{C}$, by construction

$$A'\hat{F}(S, m) = (FS + I', \widetilde{Fm \circ \Phi_{S,S}}, F\mu_{I'}),$$
$$\tilde{F}A(S, m) = (F(S + I), F\tilde{m} \circ \Phi_{S+I,S+I}, F\mu_I \circ \phi).$$

We will prove that the $\mathbf{C}$-isomorphism $\mathsf{id}_F S + \phi : FS + I' \to FS + FI$ is a monoid morphism $A'\hat{F}(S, m) \to \tilde{F}A(S, m)$, i.e., thatthe following diagrams commute, which in fact is obvious for the first one:

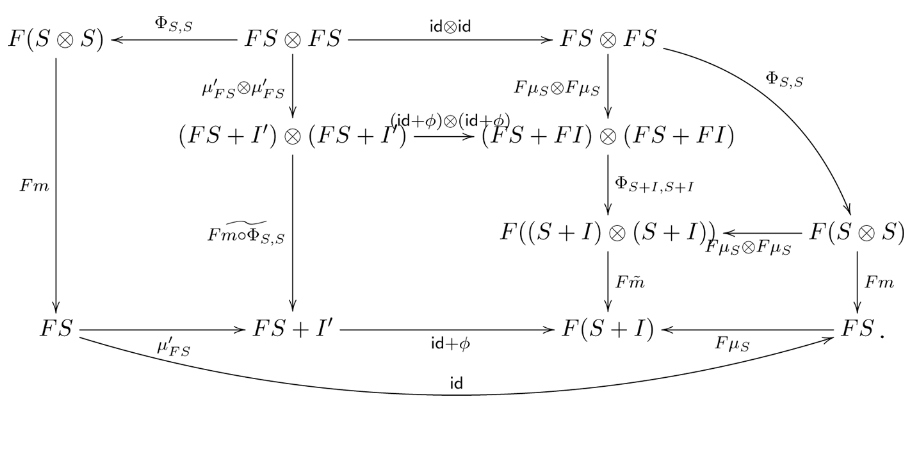




Concerning the second diagram, we need to show that precomposition with the various coproduct injections $\mu'_{FS,FS}$, $\mu'_{FS,I'}$, $\mu'_{I',FS}$, and $\mu'_{I',I'}$ equalizes both ways of the diagram.

For $\mu'_{FS,FS}$, this can be seen using the diagram below, where the left-hand cell and the bottom right rectangle commute by definition of $\tilde{\cdot}$, the top rectangle by definition of $\mathsf{id} + \phi$ and functoriality of $- \otimes -$, and the top right triangle by naturality of Φ, while the outer frame of the diagram commutes trivially.

The other cases follow analogously, except one is using coherence in order to see that the outer frame commutes. Moreover, it is clear that this isomorphism then is natural:



□

4. EXAMPLES

4.1. The Cartesian Case

Recall that the monoidal structure of a monoidal category is called *cartesian* if the tensor product is the categorical product. This includes the cases of Set, **POS**, the category of partially ordered sets and monotone maps, **Cat**, the category of small categories, every topos, and ${}_{coc,\epsilon}\mathbf{Coalg}_R$, the category of cocommutative and counital R-coalgebras with its usual tensor product (all of which are cartesian closed, such that tensoring automatically preserves binary coproducts, as required in Theorems 5 and 10). Note that this argument also applies to **CAT**, the large category of all categories, though we cannot literally speak about it as a cartesian closed category. The category **Top** of topological spaces and every bounded lattice, considered as a category, are cartesian as well (though not cartesian closed).

As we will show in the list of examples of this case to follow, we here not only find the most trivial example, that of ordinary semigroups and monoids, which motivates the (proof of) Theorem 5, but also some curious examples questioning the necessity of the hypothesis: The Example 6(2) of bounded lattices shows that the slightly inelegant assumption on the tensor functors (as compared to the condition that they preserve binary coproducts) can be of use; Example 4.1.4 (a) below shows that unitarization without our hypothesis can be possible, while Example 4.1.4 (b) below shows that it may not be.

4.1.1. Semigroups. Using as a monoidal category the cartesian category Set of sets, i.e., tensor product is cartesian product and the object I is a singleton, we get the familiar construction.

4.1.2. Topological and ordered semigroups. Since in the cartesian category **Top** of topological spaces each functor $X \times -$ preserves coproducts, one can by Theorem 5 freely adjoin a unit to every topological semigroup. The same holds for every subcategory of **Top**, closed under topological products and sums as, e.g., the category $\mathbf{Top}_2$ of Hausdorff spaces. Since **POS** is cartesian closed, one has the same result for ordered semigroups. In all of these cases, the adjunction of a unit is trivially the topologized (respectively, ordered) modification of the case of usual semigroups. In fact, these are (trivial) illustrations of the situation described in Theorem 10.

4.1.3. Semimonoidal categories. A category $\mathbf{C}$ equipped with a bifunctor $- \otimes -$, such that $(-_1 \otimes -_2) \otimes -_3 = -_1 \otimes (-_2 \otimes -_3)$, will be called a *strict semimonoidal category* (see, e.g., [1]). As a strict monoidal category is a monoid in **CAT** a strict semimonoidal category is a semigroup in **CAT**. Since **CAT** has coproducts, Theorem 5 applies and one can, to any strict semimonoidal category, freely adjoin an identity object I and so get a strict monoidal category.

One can easily extend this construction by adjoining a *strict* unit object to a nonstrict semimonoidal category. Given such $(\mathbf{C}, - \otimes -, \alpha)$ one obtains the monoidal category $(\mathbf{C} + \mathbf{1}, -\bar{\otimes}-, \bar{\alpha}, \lambda, \rho)$, where $(\mathbf{C} + \mathbf{1}, -\bar{\otimes}-, \lambda, \rho)$ are defined as above, i.e., for all C in $\mathbf{C}$ one has $\lambda_C = \rho_C : 1 \otimes C = C \xrightarrow{\mathsf{id}_C} C$, and where $\bar{\alpha}$ is the extension of α given by, e.g., $\bar{\alpha}_{1,B,C} = \big((1 \otimes B) \otimes C = B \otimes C \xrightarrow{\mathsf{id}_{B\otimes C}} B \otimes C = 1 \otimes (B \otimes C)\big)$. This construction is well known.

Since (arbitrary) monoidal categories are pseudomonoids in **CAT**, considered as a 2-category, these results lead to the question whether our process of adjoining units can be extended to pseudomonoids. This, however, would require the use of higher dimensional category theory, which is beyond the scope of this note.

4.1.4. Cosemigroups. It is well known that in the cartesian category Set of sets (as in any cartesian category), there are no nontrivial comonoids, i.e., the only comonoid on a set X is $(X, \Delta_X, !)$ with $\Delta_X : X \to X \times X$ the diagonal of X and ! the (unique) map from X to the singleton. However, there are nontrivial cosemigroups as, for example, $\mathsf{S}_1 = (\{0, 1\}, \delta)$ with $\delta(0) = \delta(1) = (0, 0)$ and $\mathsf{S}_2 = (\{0, 1, 2\}, \delta')$ with $\delta(0) = (0, 0)$, $\delta(1) = \delta(2) = (1, 1)$. It is obvious that $\mathsf{Set}^{\mathsf{op}}$ does not satisfy the second hypothesis of Theorem 5.

(a) Nevertheless, to every cosemigroup (X, δ) in Set one can cofreely adjoin a counit as follows: Put $\bar{X} = \{x \in X \mid \delta(x) = (x, x)\}$. Then $(\bar{X}, \Delta_{\bar{X}}, !)$ is a comonoid and the inclusion $\eta_X : \bar{X} \hookrightarrow X$ is a morphism of cosemigroups. If now $f : (C, \Delta_C) \to (X, \delta)$ is morphism of cosemigroups, the image $f[C]$ must be contained in $\bar{X}$, i.e., f factors (uniquely) as $C \xrightarrow{f^*} \bar{X} \xrightarrow{\eta_X} X$, and clearly f^* is a morphism of comonoids. The counitarization of S_1 is the comonoid on $\{0\}$, while the counitarization of S_2 is the comonoid on $\{0, 1\}$.
(b) If one considers instead of Set the cartesian category $\mathsf{Set}_{\neq 2}$ of all sets with cardinality $\neq 2$, our construction would not work for S_2. In fact, for this cosemigroup there is no counitarization over $\mathsf{Set}_{\neq 2}$ at all, as is easily seen.

Remark 11. The construction of the morphism $\eta_X : (\bar{X}, \Delta_{\bar{X}}) \to (X, \delta)$ in **(a)** above generalizes to any category **C** with finite products and equalizers (let simply $\eta_X : \bar{X} \to X$ be the equalizer of $\Delta_{\bar{X}}$ and δ); and this gives a counitarization of (X, δ), provided that **C** has unique factorizations of morphisms which can be lifted to **CosgrC**.

4.2. When Tensoring Preserves (Co)products Only

4.2.1. Banach algebras. By Remark 6 above we can apply Theorem 5 to the category $\mathbf{Ban}_1$ of Banach spaces and short maps. The category $\mathbf{Mon}(\mathbf{Ban}_1)$ then is the category of Banach algebras (see e.g. [30]). Thus, our construction provides the well-known (free) adjunction of a unit to a Banach algebra which does not necessarily have one.

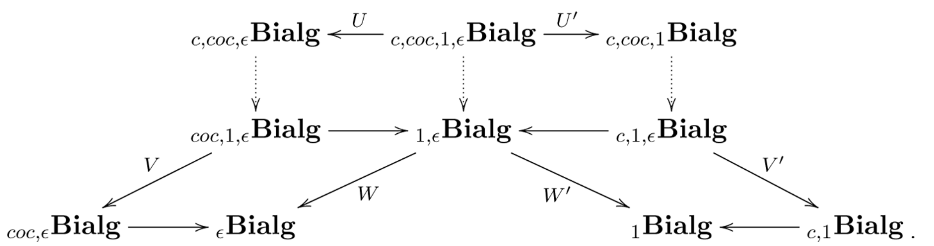
One could apply the dual argument in order to cofreely adjoin a counit to any Banach coalgebra without a counit, provided that tensoring preserves binary products, which would be the case if $\mathbf{Ban}_1$ had biproducts (see Remark 8). However, this is not so.

4.2.2. Bialgebras over a commutative unital ring. Given a commutative unital ring R, the (unital) R-bialgebras are the semigroups and monoids, respectively, in the category of R-coalgebras (with counit). This category is monoidally closed (see [32]). Hence, by Theorem 5, one can freely adjoin a unit to any nonunital counital bialgebra.

The same argument can be used with respect to the category of cocommutative R-coalgebras with counit. This possibly may be better known, since here the monoidal structure under consideration is the cartesian one (see, e.g., [12]). In addition the final statement of Theorem 5 leads to the unitarization of a nonunital commutative and cocommutative bialgebra with a counit.

Since by Lemma 9 tensoring preserves (binary) products in the category of (commutative) unital algebras, these results dualize by Theorem 7.

The following diagram displays the categories just discussed. Here all arrows are the obvious forgetful functors (embeddings) and the subscripts c (coc, 1, ϵ) refer to commutative (cocommutative, unital, counital) structures. The above statements then read as follows: The functors U, V, W admit a left adjoint, a unitarization, while the functors U', V', W' admit a right adjoint, a counitarization:



Some of these results are of particular interest in algebraic geometry. Indeed, existence of the counitarization of commutative unital biagebras implies, by the Yoneda lemma, the existence of the unitarization of semigroup schemes (over R) to obtain monoid schemes [14]. Moreover, if R is an algebraically closed field k, and if only finitely generated commutative k-algebras with a unit are considered[2], one gets unitarizations of affine algebraic semigroups to obtain affine algebraic monoids (see [21]). From the above unitarization of cocommutative counital bialgebras we obtain, again by the Yoneda lemma, the unitarization of a formal semigroup to get a formal monoid (see [12]).

4.2.3. Categories without identities. Here we use the categories **Grph** of (directed) graphs and its (non-full) subcategory of O-**Grph** of graphs with a fixed set O as set of vertices (and identity-on-objects graph homomorphisms) from [27, pp. 48–49]. The former category is symmetric monoidal for the pullback over O along the source and target maps, the unit object being the O-graph on O with source and target maps the identities.

Roughly speaking, a *semicategory* (also called a *taxonomy*, see [2]) is a category where objects do not necessarily have identities (in particular, the source and target maps are not required to be onto). A semigroup (respectively, monoid) in O-**Grph**, thus, is a (small) semicategory (respectively, category) with object set O. A paradigmatic example of a semicategory is given by Hilbert spaces with Hilbert–Schmidt operators as morphisms, since the identity on an infinite dimensional Hilbert space fails to be a Hilbert–Schmidt operator (see, e.g., [37]).

Since the category O-**Grph** has finite coproducts (a binary coproduct here is essentially the disjoint sum of the sets of edges), which are preserved by the pullback,

[2]Observe that these are closed under tensor product and binary product.

we may apply Theorem 5 to freely add identities to a (small) semicategory to get a (small) category (both with object set O). The identities can, by construction, be identified with the elements of O.

This construction is of a local nature since the category obtained from a semicategory is universal only among those categories with object set O. It may be globalized as follows. Let **Semi-Cat** denote the category of all *semicategories* with object class all objects from **Sgr**(O-**Grph**) for all sets O, and with arrows the obvious ones, the *semifunctors*. We then have (see [2] for quite a different proof).

Proposition 12. *The forgetful functor* $|-|$: **Semi-Cat** $\to$ **Cat** *has a left adjoint.*

Proof. Let S be a (small) semicategory (and let $O(\mathsf{S})$ be its object set). Let C be a category, and let $F = (F_O, F_A) : \mathsf{S} \to |\mathsf{C}|$ be a semifunctor (F_O is the object function, while F_A is the arrow function of F). Let $\tilde{\mathsf{S}}$ be the unitarization of S in $O(\mathsf{S})$-**Grph**. Then, we define a functor $F^{\#} = (F_O, F'_A) : \tilde{\mathsf{S}} \to \mathsf{C}$ by $F'_A(f) = F_A(f)$ for every arrow f in S, and $F'_A(\mathsf{id}_x) = \mathsf{id}_{F_O(x)}$ for every object $x \in O(\mathsf{S})$. This functor is the unique functor $G : \tilde{\mathsf{S}} \to \mathsf{C}$ with the property that $G_A(f) = F_A(f)$ for every arrow f of S and $G_O(x) = F_O(x)$ for every object x of $O(\mathsf{S})$. □

Remark 13. The above constructions, both the adjunction of identities to semicategories, and the globalization of its universal property, may be extended to internal (semi)categories [8] in a locally cartesian closed category **C** (see, e.g., [42]) with binary coproducts, as it is clear by the following somewhat more abstract description of this construction.

Consider the slice category $\mathsf{Set} \downarrow O$ with its forgetful functor $|-|$ into Set. O-**Grph** is (by the identification $(A, c, d) = ((A, c), (A, d))$ isomorphic to) the subcategory of the (non-full) category $(\mathsf{Set} \downarrow O)^2$ with those objects $((A, f), (B, g))$ with $A = B$ (f and g are the source and target maps) and those morphisms (ϕ, ψ) with $\phi = \psi$.

Since, as for any slice category, the functor $|-|$ creates coproducts as $(A, f) + (B, g) = (A + B, [f, g])$ (here $[f, g]$ is the morphism $A + B \to O$ with components f and g) and, as for any product category, coproducts are constructed coordinatewise, one has in $(\mathsf{Set} \downarrow O)^2$

$$\begin{aligned}((A, f), (B, g)) + ((A', f'), (B', g')) &= ((A, f) + (A', f')), ((B, g) + (B', g')) \\ &= ((A + A', [f, f']), (B + B', [g, g'])).\end{aligned}$$

This shows in particular, that O-**Grph** is closed in $(\mathsf{Set} \downarrow O)^2$ under coproducts.

Since the pullback of $A \xrightarrow{c} O$ along $B \xrightarrow{t} O$ is the product $(A, c) \times (B, t)$ in $\mathsf{Set} \downarrow O$, the tensor product in O-**Grph** defined above can alternatively be described as

$$(A, c, d) \otimes (B, s, t) = (|(A, c) \times (B, s)|, s \circ \pi_B, d \circ \pi_A),$$

where π_A and π_B are the projections of the pullback in question.

It is now easy to see that, in O-**Grph**, tensoring preserves binary coproducts since, in $\mathsf{Set} \downarrow O$, product multiplication preserves binary coproducts.

Thus, the construction above can be generalized to internal categories in any locally cartesian closed category with binary coproducts. Every quasi-topos satisfies these conditions by definition (see, e.g., [31]) and, thus, in particular the categories of Kowalsky's convergence spaces, Choquet's pseudo-topologies, or Spanier's quasi-topologies.

4.3. When Tensoring Preserves Coproducts and Products

4.3.1. Rings. This example is a special instance of Section 4.3.3 below. We only mention it here already because of its generally known importance.

Since (not necessary unital) rings are semigroups in the monoidal category **Ab** of abelian groups with its canonical tensor product, our construction gives the free adjunction of a unity to a ring, which is not necessarily unital. While the familiar construction (see, e.g., [22, 2.27]) also requires to pay attention to the additive structure of the ring when adjoint the unity, this comes for free here. Note, however, that our abstract construction nevertheless directly translates into the familiar one.

Remark 14. This example moreover shows, that the universal adjunction of a unit—being obviously a minimal monoid extension in the case of semigroups—in general will fail to be so: The nonunital ring $2\mathbb{Z}$ of even integers admits an embedding into the unital ring $\mathbb{Z}$, which is not isomorphic to the universal construction and properly contained in it.

4.3.2. Semirings. The category ${}_c\mathbf{Mon}$ of (ordinary) commutative monoids is known to be monoidally closed (see Remark 6). The tensor product, thus, constructed as in **Ab** and the internal hom functor is given by the bi-additive maps. Thus, the semigroups in this monoidal category are semirings (without unit), while the monoids are the unital semirings (whose additive identity is not necessarily absorbing, as it is sometimes required in the definition of a semiring). Theorem 5 thus applies.

Moroever, ${}_c\mathbf{Mon}$ has binary biproducts and tensor functors are additive. Thus, Theorem 7 applies as well. Calling a comonoid in ${}_c\mathbf{Mon}$ a semi-coring (see [1] for a related notion), we therefore conclude, that one can cofreely adjoin a counit to any semi-coring, which does not necessarily have a counit.

4.3.3. Algebras and coalgebras over a commutative unital ring. Let R be a commutative unital ring. Then the same considerations as for rings apply, except one considers, instead of **Ab**, the category $\mathbf{Mod}_R$ with its canonical tensor product. We thus get a left adjoint of the forgetful functor from the category ${}_1\mathbf{Alg}_R$ of unital R-algebras to the category $\mathbf{Alg}_R$ of R-algebras which are not necessarily unital.

Since $\mathbf{Mod}_R^{\mathsf{op}}$, the dual of the category of R-modules, is monoidal as well, the categories $\mathbf{Mon}(\mathbf{Mod}_R^{\mathsf{op}})$ and $\mathbf{Sgr}(\mathbf{Mod}_R^{\mathsf{op}})$ are defined and one has $\mathbf{Mon}(\mathbf{Mod}_R^{\mathsf{op}}) = {}_\epsilon\mathbf{Coalg}_R^{\mathsf{op}}$, the category of R-coalgebras with a counit, and $\mathbf{Sgr}(\mathbf{Mod}_R^{\mathsf{op}}) = \mathbf{Coalg}_R^{\mathsf{op}}$, the category of coalgebras which do not necessarily have a counit. By Remark 8, we can apply Theorem 7, that is, one can adjoin a counit to any R-coalgebra, and *cofreely* so.

Moreover, the dualization functor $F = (-)^\star : \mathbf{Mod}_R^{\mathsf{op}} \to \mathbf{Mod}_R$ is a lax monoidal functor (the canonical morphisms $M^\star \otimes M^\star \to (M \otimes M)^\star$ form a natural transformation Φ, and the canonical morphism $\phi : R \to R^\star$ here is even an isomorphism). We thus get induced functors $\hat{F} : \mathbf{Coalg}_R^{\mathsf{op}} \to \mathbf{Alg}_R$ and $\tilde{F} : {}_\epsilon\mathbf{Coalg}_R^{\mathsf{op}} \to {}_1\mathbf{Alg}_R$ by Proposition 3, such that we have the commutative diagram

$$\begin{array}{ccc} {}_\epsilon\mathbf{Coalg}_R^{\mathsf{op}} & \xrightarrow{\tilde{F}} & {}_1\mathbf{Alg}_R \\ {\scriptstyle V'}\downarrow & & \downarrow{\scriptstyle V} \\ \mathbf{Coalg}_R^{\mathsf{op}} & \xrightarrow[\hat{F}]{} & \mathbf{Alg}_R \end{array}$$

where V and V' have left adjoints A and A', respectively. The functors $\tilde{F}$ and $\hat{F}$ are the so-called *dual algebra functors* (note that usually only $\tilde{F}$ is considered).

Clearly, the dualization functor $(-)^\star$ preserves binary coproducts as well, such that the assumptions of Theorem 10 are satisfied as well. We thus get the following proposition.

Proposition 15. *Let R be a unital commutative ring. Then, for every R-coalgebra C, the unitarization of the dual algebra of C coincides with the dual algebra of the counitarization of C.*

It is known that the dual algebra functor $\tilde{F} : {}_\epsilon\mathbf{Coalg}_R \to {}_1\mathbf{Alg}_R$ has a left adjoint. For $R = k$ a field, this is given by the so-called *finite dual* of an algebra. For arbitrary rings, this has been shown in [33], and it is clear from that proof that also the dual algebra functor $\hat{F} : \mathbf{Coalg}_R \to \mathbf{Alg}_R$ has a left adjoint. Let us call these left adjoints "finite duals" as well. Then, by composition of adjoints, we obviously get a kind of inverse of the above result as follows.

Proposition 16. *Let R be a unital commutative ring. Then, for every R-algebra A, the counitarization of the finite dual of A coincides with the finite dual of the unitarization of A.*

4.3.4. Differential algebras and coalgebras. Let R be a commutative unital ring. A *derivation* ∂ on R is an endomorphism of the underlying additive group of R satisfying Leibniz' rule, i.e., for every $a, b \in R$,

$$\partial(ab) = a\partial(b) + \partial(a)b.$$

The pair (R, ∂) is called a *commutative differential unital ring* [24]. Every commutative unital ring R may be equipped with the trivial derivation 0.

An R-module M is said to be an *(R, ∂)-module*, if it is equipped with a *(module) derivation* D, i.e., an endomorphism of the underlying additive group of M such that for every $a \in R$ and $x \in M$,

$$D(ax) = aD(x) + \partial(a)x.$$

The pair (M, D) is called a *differential module over* (R, ∂). If $\partial = 0$, then D is just a R-linear map. A morphism of differential modules over (R, ∂) is an R-linear map, commuting with the derivations. This defines the category $\mathbf{diffMod}_{(R,\partial)}$. The map $M \to (M, 0)$ evidently defines an embedding of $\mathbf{Mod}_R$ into $\mathbf{diffMod}_{(R,0)}$.

The category $\mathbf{diffMod}_{(R,\partial)}$ has biproducts, constructed as follows: If (M, D_M) and (N, D_N) are differential modules over (R, ∂), then so is $(M \oplus N, D_{M,N})$ with $D_{M,N}$ defined by $(D_M + D_N)(x, y) = (D_M(x), D_N(y))$, $x \in M, y \in N$.

The category $\mathbf{diffMod}_{(R,\partial)}$ also admits a symmetric monoidal structure as follows. If $(M, D_M), (N, D_N)$ are differential modules over (R, ∂), let $D : M \times N \to M \otimes N$ be the bi-additive map with $D(x, y) = D_M(x) \otimes y + x \otimes D_N(y)$. This map satisfies $D(ax, y) = aD(x, y) + \partial(a)(x \otimes y) = D(x, ay)$ $(a \in R, x \in M, y \in N)$ and, thus, induces a group endomorphism $D_M \otimes \mathsf{id}_N + \mathsf{id}_M \otimes D_N$ of $M \otimes N$, which is easily seen to be module derivation. Now define $(M, D_M) \otimes (N, D_N) := (M \otimes N, D_M \otimes \mathsf{id}_N + \mathsf{id}_M \otimes D_N)$. (R, ∂) is the left and right unit of this tensor product.

The categories $\mathbf{Sgr}(\mathbf{diffMod}_{(R,\partial)})$ $(\mathbf{Mon}(\mathbf{diffMod}_{(R,\partial)}))$ are called the categories of (*unital*)*differential* (R, ∂)*-algebras*. In particular, a commutative differential unital ring is a commutative monoid in $\mathbf{diffMod}_{(\mathbb{Z},0)}$.

Dually, the categories $\mathbf{Cosgr}(\mathbf{diffMod}_{(R,\partial)})$ $(\mathbf{Comon}(\mathbf{diffMod}_{(R,\partial)}))$ are called the categories of (*counital*) *differential* (R, ∂)*-coalgebras*. A (counital) differential (R, ∂)-coalgebra, thus, is a pair (C, D), where C is a (counital) R-coalgebra and D is a *coderivation*, i.e., a module derivation (of the underlying module) of C satisfying the *co-Leibniz rule* $(D \otimes id + id \otimes D) \circ \Delta = \Delta \circ D$, where Δ is the comultiplication of C [35].

From the preservation by $M \otimes -$ of binary biproducts on $\mathbf{Mod}_R$, it follows that $(M, D) \otimes -$ also preserves binary biproducts on $\mathbf{diffMod}_{(R,\partial)}$, and thus we may freely add a unit to a differential (R, ∂)-algebra to obtain a unital differential (R, ∂)-algebra, and cofreely add a counit to a differential (R, ∂)-coalgebra to obtain a counital differential (R, ∂)-coalgebra.

4.3.5. Monoids and comonoids in Hilbert spaces. The category **Hilb** of Hilbert spaces (real or complex) and linear bounded operators has binary biproducts (see [17, 19]). It is a symmetric monoidal category by the usual hilbertian tensor product (see [23]) and each tensor functor $H \otimes -$ is additive and preserves binary biproducts (see [37]); **Hilb** carries a contravariant involutive endofunctor $(-)^\dagger$ defined by $(H \xrightarrow{f} H')^\dagger := H' \xrightarrow{f^\dagger} H$, where $f^\dagger$ is the adjoint of f. Denoting by $\mathbf{Hilb}^{\mathsf{op}}$ the dual monoidal category of **Hilb**, the functor $(-)^\dagger : \mathbf{Hilb}^{\mathsf{op}} \to \mathbf{Hilb}$ is a strict monoidal functor.

By Proposition 3 and Remark 4, the functor $(-)^\dagger$ thus induces an *adjoint monoid functor* $M : (\mathbf{Comon}(\mathbf{Hilb}))^{\mathsf{op}} \to \mathbf{Mon}(\mathbf{Hilb})$ as well as an *adjoint comonoid functor* $C : \mathbf{Mon}(\mathbf{Hilb}) \to (\mathbf{Comon}(\mathbf{Hilb}))^{\mathsf{op}}$ and, analogously, an *adjoint semigroup functor* and an *adjoint cosemigroup functor*. The functor C then is a left adjoint, in fact an equivalence inverse, of M.

Using the same arguments as in Section 4.3.3 (replacing $(-)^\star$ by $(-)^\dagger$), one now gets the following result where, contrary to the case of Banach algebras, every cosemigroup (H, m) lives on the same space H as its adjoint semigroup $(H, m)^\dagger = (H, m^\dagger)$ (and similar for comonoids).

Proposition 17.

1. *To every semigroup in* **Hilb**, *one can freely adjoin a unit.*
2. *To every cosemigroup in* **Hilb** *one can cofreely adjoin a counit.*
3. *For every cosemigroup in* **Hilb** *the unitarization of its adjoint semigroup coincides with the adjoint monoid of its counitarization.*
4. *For every semigroup in* **Hilb** *the counitarization of its adjoint cosemigroup coincides with the adjoint monoid of its unitarization.*

Note that we do not use the term *Hilbert algebra* here, because that term already is in use in (more than one) different meanings. Examples (and counterexamples) of the structures discussed above are, e.g., as follows:

1. Every semigroup (respectively, monoid) in **Hilb** is a Banach algebra (respectively, unital Banach algebra) up to a change of an equivalent Banach norm (if (H, m) is a semigroup, then because the bilinear map m_0 corresponding to m is continuous, one has, for every $x, y \in H$, $\|m_0(x, y)\| \leq C\|x\|\|y\|$ for some positive real number C, and from [36] it is known that we may find an equivalent norm $\|\cdot\|'$ on H such that $\|m_0(x, y)\|' \leq \|x\|'\|y\|'$; moreover, if H has a unit $e \neq 0$, we may even choose the equivalent norm so that $\|e\|' = 1$).
2. Conversely, not every Banach algebra which is also a Hilbert space (and with the norm induced by its inner product) is a semigroup (or monoid) in **Hilb**. Indeed, the Hilbert tensor product does not coincide in general with the projective tensor product. In particular, a continuous bilinear mapping induces a bounded linear mapping on the hilbertian tensor product if, and only if, it is a weakly Hilbert–Schmidt mapping (see [23]). Simple computations show that the multiplication in a full matrix algebra (see [4, p. 367]) of an infinite order, while bilinear and continuous, is not a weakly Hilbert–Schmidt mapping and, thus, such a Banach algebra (which is also a Hilbert space) is not a semigroup object in **Hilb**. For the same reason the algebra of Hilbert–Schmidt operators [16, 39] on an infinite-dimensional Hilbert space is not a semigroup object in **Hilb**.
3. For any set I, the algebra $\ell^2(I)$ (under component-wise operations, see [4]) is a semigroup in **Hilb** (and even a monoid when I is finite).
4. Using the Peter–Weyl theorem [20, p. 24] we can check that for any compact group G the L^2-algebra $L^2(G)$ of G (the space of square-integrable real or complex-valued functions with respect to the *normalized* Haar measure of G and with the convolution product, see again [4]) also is a semigroup in **Hilb**. This algebra admits a unit, and so is a monoid in **Hilb**, when G is a finite group.

4.3.6. *R*-rings and *R*-corings. There is no generally accepted notion of an algebra over a noncommutative unital ring R. The most feasible generalizations from commutative rings to this situation seem to be the concepts of unital *R-rings* in the sense of Cohn [11] and of R-rings without a unit in the sense of [7]. These then form the categories $\mathbf{Mon}({}_R\mathbf{Mod}_R)$ and $\mathbf{Sgr}({}_R\mathbf{Mod}_R)$, respectively. Recall that, in general, the monoidal structure of ${}_R\mathbf{Mod}_R$ fails to be symmetric, but is left and right closed, such that the assumptions of Theorem 5 are satisfied. Then the same considerations as in the previous example allow for a free adjunction of a unit to an R-ring to obtain a unital R-ring.

Moreover, Theorem 7 applies by Remark 8. Thus, one can cofreely adjoin a counit to any R-coring in order to obtain a counital R-coring (see [10, 41] for the definition of an R-coring).

4.3.7. Graded semigroups and monoids. As a further application, we describe a graded version of our results, which is of interest for algebraic topology (a filtered version will be investigated later). See [28] for the proofs concerning graded categories.

Let $\mathbb{C} = (\mathbf{C}, -\otimes -, I)$ be a monoidal category with arbitrary coproducts, preserved by all functors $C \otimes -$ and $- \otimes C$. Moreover, let M be a (usual) monoid with underlying set M and unit e_M. Then there is a monoidal category $\mathbf{Gr}(\mathbb{C})_\mathsf{M} = (\mathbf{C}^M, -\otimes -, Y)$ of M-*graded objects of* $\mathbf{C}$ where the following statements hold:

1. $\mathbf{C}^M$ is the M-fold power of the category $\mathbf{C}$, i.e., objects are families $(F_x)_{x\in M}$ of objects of $\mathbf{C}$, and a morphism $f: F \to G$ is a family $(f_x)_{x\in M}$ of morphisms $f_x : F_x \to G_x$ in $\mathbf{C}$. Composition and identities in $\mathbf{C}^M$ are taken coordinate-wise.
2. $F \otimes G$ is defined by

$$(F \otimes G)_{z\in M} = \coprod_{xy=z} F_x \otimes G_y.$$

3. Y is defined as

$$Y_x = \begin{cases} 0 \text{ if } x \neq e_\mathsf{M} \\ I \text{ if } x = e_\mathsf{M}, \end{cases}$$

where 0 is the initial object of $\mathbf{C}$.

$\mathbf{C}^M$ has coproducts, defined pointwise by

$$\Big(\coprod_{j\in J} F^j\Big)_{x\in M} = \coprod_{j\in J} F^j_x,$$

and these are preserved by all functors $F \otimes -$ and $- \otimes F$. Hence, we obtain by Theorem 5.

Theorem 18. *Let $\mathbb{C} = (\mathbf{C}, -\otimes -, I)$ be a monoidal category such that $\mathbf{C}$ has binary coproducts, where all functors $C \otimes -$ and $- \otimes C$ preserve coproducts of the form $S + I$, and let M be a monoid. Then the forgetful functor $\mathbf{SgrGr}(\mathbb{C})_\mathsf{M} \to \mathbf{MonGr}(\mathbb{C})_\mathsf{M}$ has a left adjoint.*

Applications of this graded generalization of (co)unitarization concern for instance the following cases:

- For $\mathbb{C}$, the cartesian category $\mathbf{Set}$ of sets, we get graded semigroups and graded monoids, respectively (i.e., usual semigroups (S, m) and monoids (S, m, e) that may be written as a set-theoretic disjoint sum $S = \coprod_{x\in M} S_x$ such that $m(a, b) \in S_x$ for every $a \in S_y$, $b \in S_z$ and $x = yz$, $e \in S_{e_\mathsf{M}}$). (See [38] for a slightly less general notion.)

- For $\mathbb{C} = \mathbf{Mod}_R$ (with its usual tensor product), we get the usual M-graded R-algebras with or without a unit (see [26, p. 177] for this description), which might be better known in the cases $\mathsf{M} = \mathbb{Z}$ and $\mathsf{M} = \mathbb{N}$ (graded in positive degrees), seen as additive monoids (see e.g. [29]). When $\mathsf{M} = \mathbb{Z}/2\mathbb{Z}$, we recover superalgebras [13]. As another example, the monoid algebra $R[\mathsf{M}]$ of a monoid M is obviously an M-graded R-algebra.
- If $\mathbb{C} = \mathbf{Mod}_R^{\mathrm{op}}$, we get M-graded coalgebras with or without a counit.

4.4. Extensions

4.4.1. *-algebras. This example is devoted to the problem of lifting our construction to *-monoids.

Recall that a *ring with involution* is a (commutative and unital) ring R together with a self inverse automorphism ϕ of R. This automorphism induces an involutive automorphism $(-)^\phi$ on the category $\mathbf{Alg}_R$ by assigning to an algebra $\mathsf{A} = (A, (\lambda \cdot -)_{\lambda \in R}, m)$ (where A is an abelian group, $\lambda \cdot -$ denotes a scalar multiplication making A into an R-module, and m is a bilinear and associative multiplication) the algebra $\mathsf{A}^\phi = (A, (\phi\lambda \cdot -)_{\lambda \in R}, m^{\mathrm{op}})$. If ϕ is the identity, A^ϕ is simply the opposite algebra A^{op} of A. The other interesting case is that of complex algebras with ϕ being the complex conjugation; in this case, we will write $\bar{\mathsf{A}}$ instead of A^ϕ.

Definition 19. Let (R, ϕ) be a ring with involution. A **-algebra over* (R, ϕ) is an R-algebra A equipped with an involutive algebra homomorphism $\star : A \to A^\phi$. A homomorphism of *-algebras then is an algebra homomomorphism commuting with the $\star$-operations.

Note that a *-algebra over (R, ϕ) thus is a functor algebra (and a functor coalgebra) over $\mathbf{Alg}_R$ with respect to the functor $(-)^\phi$. Thus, the category $^\star\mathbf{Alg}_{(R,\phi)}$ of *-algebras over (R, ϕ) is the full subcategory of $\mathbf{Alg}(-)^\phi$ and $\mathbf{Coalg}(-)^\phi$, respectively, spanned by those (co)algebras satisfying $\star \circ \star = id$.

The category $^\star\mathbf{Alg}_{(\mathbb{Z},\mathrm{id})}$ is called the *category of *-rings* (see also [5, 18], where such structures are called *rings with involution*).

Note that, if A is a (real or complex) Banach algebra, so are A^{op} and $\bar{A}$, equipped with the given norm. We thus have, by obvious generalization, categories $^\star\mathbf{Ban}_1$ of real and of complex *-Banach algebras as subcategories of categories of functor (co)algebras over $\mathbf{Sgr}(\mathbf{Ban}_1)$.

Some of the structures discussed here are also known as *$B^\star$-algebras* (see, e.g., [36], where also the problem of adjoining a unit is discussed).

With notation just introduced a $C^\star$-algebra is a complex *-Banach algebra A satisfying the *$C^\star$-identity*

$$\forall\, x \in A,\;\; \|xx^\star\| = \|x\|\|x^\star\|.$$

A *homomorphism* f between $C^\star$-algebras is an algebra homomorphism commuting with the operation $(-)^\star$, which is bounded (equivalently, which is a short map).

Thus, $C^\star$-algebras form a (non full) subcategory of the category $^\star\mathbf{Ban}_1$ as defined above.

The various categories of ⋆-algebras defined above clearly come in unital and non-unital versions. It is therefore worthwhile to consider the problem of adjoining a unit to a nonunital ⋆-algebra. We here use the notations $^\star\mathbf{Sgr}\mathbb{C}$ and $^\star\mathbf{Mon}\mathbb{C}$ in the obvious way, that is, we assume that there are given self inverse functors $(-)^\phi$ on $\mathbf{Sgr}\mathbb{C}$ and $\mathbf{Mon}\mathbb{C}$, respectively, such that $(-)^\phi \circ V = V \circ (-)^\phi$ (with V as in Section 2).

Proposition 20. *Let $\mathbb{C}$ be a monoidal category, such that one can freely adjoin a unit to any semigroup over $\mathbb{C}$. Let $(-)^\phi : \mathbf{Sgr}\mathbb{C} \to \mathbf{Sgr}\mathbb{C}$ be a self-inverse automorphism on $\mathbf{Sgr}\mathbb{C}$, which can be restricted to an automorphism on $\mathbf{Mon}\mathbb{C}$. Then the adjunction between $\mathbf{Sgr}\mathbb{C}$ and $\mathbf{Mon}\mathbb{C}$ can be lifted to an adjunction between $^\star\mathbf{Sgr}\mathbb{C}$ and $^\star\mathbf{Mon}\mathbb{C}$.*

Proof. Let S be a semigroup and $\star : \mathsf{S} \to \mathsf{S}^\phi$ its ⋆-operation. If $\eta_\mathsf{S} : \mathsf{S} \to \tilde{\mathsf{S}}$ is the unit of the given adjunction, there is a unique monoid morphism $\tilde{\star} : \tilde{\mathsf{S}} \to \widetilde{\mathsf{S}^\phi}$ such that the diagram

$$\begin{array}{ccc} \mathsf{S} & \xrightarrow{\eta_\mathsf{S}} & \tilde{\mathsf{S}} \\ {\scriptstyle\star}\downarrow & & \downarrow{\scriptstyle\tilde{\star}} \\ \mathsf{S}^\phi & \xrightarrow[\eta_{\mathsf{S}^\phi}]{} & \widetilde{\mathsf{S}^\phi} \end{array}$$

commutes, and this is, by construction, self-inverse. It now suffices to observe that by composition and uniqueness of adjoints $\tilde{\mathsf{S}}^\phi \xrightarrow{\eta_{\mathsf{S}^\phi}} \widetilde{\mathsf{S}^\phi} = \tilde{\mathsf{S}}^\phi \xrightarrow{(\eta_\mathsf{S})^\phi} \tilde{\mathsf{S}}^\phi$ (the functors $(-)^\phi$ are isomorphisms on $\mathbf{Mon}\mathbb{C}$ and $\mathbf{Sgr}\mathbb{C}$, respectively).

It is then easy to see that $\eta_\mathsf{S} : (\mathsf{S}, \star) \to (\tilde{\mathsf{S}}, \tilde{\star})$ is a universal morphism for the forgetful functor $^\star\mathbf{Sgr}\mathbb{C} \to {}^\star\mathbf{Mon}\mathbb{C}$. □

Thus, the free adjunction of units is possible for ⋆-rings, ⋆-algebras, ⋆-Banach algebras, and $^\star R$-rings.

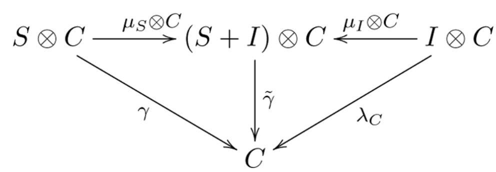
Concerning $C^\star$-algebras, it would remain to show that the $C^\star$-identity holds for $A + \mathbb{C}$. But this, unfortunately is not the case, as it is well known. For this one has to change the norm of $A + \mathbb{C}$ to an equivalent one (see [15]).

4.4.2. Monoid and semigroup actions. Given a symmetric monoidal category $\mathbb{C} = (\mathbf{C}, - \otimes -, I)$ and a monoid $\mathsf{M} = (M, m, e)$ in $\mathbb{C}$, there is the well known category $_\mathsf{M}\mathbf{Lact}$ of left M-(monoid) actions (see [27]). Left M-actions thus are pairs (C, γ) with C an object and $\gamma : M \otimes C \to C$ a morphism in $\mathbf{C}$ satisfying the usual axioms concerning associativity and unit. In the same way, one can define left actions of a semigroup $\mathsf{S} = (S, m)$ in $\mathbb{C}$ by dropping the condition on the action by the unit and so gets categories $_\mathsf{S}\mathbf{Lact}$ of left S-(semigroup) actions.

Monoid actions on a set in the usual sense are monoid actions in the cartesian category Set, while for a monoid M in the monoidal category $\mathbf{Ab}$ of abelian groups (with the usual tensor product), i.e., a unital ring, the category $_\mathsf{M}$Lact is the category $_\mathsf{M}\mathbf{Mod}$ of left M-modules.

The following example is well known for actions in Set: A semigroup action of a semigroup S on a set X can be made into monoid action by adjoining an identity to the semigroup and requiring that it acts as the identity transformation on X. Since every monoid action of this monoid, by restricting its action to S, induces a semigroup action of S, and these operations are mutually inverse one gets an isomorphism of the categories ${}_{\mathsf{S}}\mathbf{Lact}$ and ${}_{\tilde{\mathsf{S}}}\mathbf{Lact}$.

The process just described for Set can in fact be generalized to any symmetric monoidal category $\mathbb{C}$ satisfying the hypotheses of Theorem 5 as follows: If $\mathsf{S} = (S, m)$ is a semigroup in $\mathbb{C}$, $\tilde{S} = (S + I, \tilde{m}, \tilde{e})$ its unitarization, and (C, γ) an S-action, let $\tilde{\gamma} : (S + I) \otimes C \to C$ be the unique morphism such that the diagram



commutes. Then, using the methods of the proofs in Section 3, it is easy (though somewhat lengthy) to see that $(C, \tilde{\gamma})$ is an $\tilde{\mathsf{S}}$-action, and that the assignment $(C, \gamma) \mapsto (C, \tilde{\gamma})$ defines an isomorphism of categories

$$ {}_{\mathsf{S}}\mathbf{Lact} \simeq {}_{\tilde{\mathsf{S}}}\mathbf{Lact}. \tag{10}$$

We can avoid this cumbersome argument by directly applying Theorem 5 in case $\mathbb{C}$ is a symmetric monoidal closed category. Here we can use an equivalent description of an M-action (C, γ), familiar from the cases $\mathbb{C} = \mathsf{Set}$ or $\mathbb{C} = \mathbf{Mod}_R$, as a monoid morphism from M into the endomorphism monoid of C.

By elementary facts on symmetric monoidal categories (see, e.g., [9, 6.1.]), for every $\mathbf{C}$-object C one has a monoid ${}_{1}\mathsf{End}(C) = ([C, C], c_{C,C,C}, e_C)$ in $\mathbb{C}$, where $[-, -]$ denotes the internal hom functor, $c_{C,C,C}$ is the composition morphism $[C, C] \otimes [C, C] \to [C, C]$, and $e_C : I \to [C, C]$ is the unit morphism (corresponding by adjunction to id_C). We call, for obvious reasons, ${}_{1}\mathsf{End}(C)$ the *endomorphism monoid* of C and $\mathsf{End}(C) = ([C, C], c_{C,C,C})$ the *endomorphism semigroup* of C.

We will also use the *evaluation morphism* $ev_C : [C, C] \otimes C \to C$, corresponding by adjunction to the identity of $[C, C]$. Note the equation

$$C \simeq I \otimes C \xrightarrow{e_C \otimes C} [C, C] \otimes C \xrightarrow{ev_C} C = \mathsf{id}_C\ . \tag{11}$$

By adjunction, there corresponds to any $\mathbb{C}$-morphism $X \otimes C \to C$ a $\mathbf{C}$-morphism $X \to [C, C]$. We will use the following standard categorical observation.

Fact 21. If M is a monoid and C an object in $\mathbf{C}$, for any morphism $M \otimes C \xrightarrow{\gamma} C$ and its mate $M \xrightarrow{\gamma^*} [C, C]$, corresponding to γ by adjunction, the following statements are equivalent:

1. (C, γ) is a left M-action;
2. $\mathsf{M} \xrightarrow{\gamma^*} {}_{1}\mathsf{End}(C)$ is a monoid morphism.

For left M-actions (C, γ) and (D, δ) and a $\mathbf{C}$-morphism $f : C \to D$ the following statements are equivalent:

3. $f : (C, \gamma) \to (D, \delta)$ is a morphism of left M-actions;
4. The following diagram commutes:

$$\begin{array}{ccccc} M \otimes C & \xrightarrow{\gamma^* \otimes C} & [C, C] \otimes C & \xrightarrow{ev_C} & C \\ {\scriptstyle M\otimes f}\downarrow & & & & \downarrow{\scriptstyle f} \\ M \otimes D & \xrightarrow{\delta^* \otimes D} & [D, D] \otimes D & \xrightarrow{ev_D} & C\,. \end{array}$$

Similarly for semigroups and their actions.

We now can prove the following proposition, using these equivalent description of actions and their morphisms.

Proposition 22. *Let $\mathbb{C}$ be a symmetric monoidal closed category with binary coproducts. Then, for any semigroup S in $\mathbb{C}$ with unitarization $\tilde{\mathsf{S}}$, the categories ${}_{\mathsf{S}}Lact$ of left S-semigroup actions and ${}_{\tilde{\mathsf{S}}}Lact$ of left $\tilde{\mathsf{S}}$-monoid actions are (concretely) isomorphic.*

Proof. If $(C, S \otimes C \xrightarrow{\gamma} C)$ is an S action, there exists, by Theorem 5, a unique monoid morphism $\tilde{\gamma} := (\gamma^*)^\sharp : \tilde{\mathsf{S}} \to {}_{1\mathsf{End}}(C)$ with $\tilde{\gamma} \circ \mu_S = \gamma^*$. Thus, one can assign to every ${}_{\mathsf{S}}\mathbf{Lact}$-object a ${}_{\tilde{\mathsf{S}}}$ $\mathbf{Lact}$-object by $(C, \gamma) \mapsto (C, \tilde{\mathsf{S}} \xrightarrow{\tilde{\gamma}}_1 \mathsf{End}(C)) =: \Psi(C, \gamma)$. In order to see that this, with $\Psi(f) = f$, becomes a functor, we need to show that the right-hand rectangle in the diagram below commutes, which will be the case, provided that precomposition with $\mu_S \otimes C$ and $\mu_I \otimes C$ equalizes both ways:

$$\begin{array}{ccccccc} & & & \gamma^*\otimes C \ (S\otimes C \to [C,C]\otimes C) & & & \\ S \otimes C & \xrightarrow{\mu_S \otimes C} & (S + I) \otimes C & \xrightarrow{\tilde{\gamma} \otimes C} & [C, C] \otimes C & \xrightarrow{ev_C} & C \\ {\scriptstyle S\otimes f}\downarrow & & \downarrow{\scriptstyle (S+I)\otimes f} & & & & \downarrow{\scriptstyle f} \\ S \otimes D & \xrightarrow{\mu_S \otimes D} & (S + I) \otimes D & \xrightarrow{\tilde{\delta} \otimes D} & [D, D] \otimes D & \xrightarrow{ev_D} & C\,. \\ & & & \delta^*\otimes C \ (S\otimes D \to [D,D]\otimes D) & & & \end{array}$$

For μ_S this follows, by definition of $\tilde{\gamma}$ and $\tilde{\delta}$, from commutativity of the left-hand cell and the fact that f is a morphism in ${}_{\mathsf{S}}\mathbf{Lact}$.

Concerning precomposition with μ_I recall this is the unit of $\tilde{\mathsf{S}}$ and, thus, preserved by the monoid morphisms $\tilde{\gamma}$ and $\tilde{\delta}$, which, by definition, is equivalent to saying $\tilde{\gamma} \circ \mu_I = e_C$ and $\tilde{\delta} \circ \mu_I = e_D$. This reduces the required equality to $f \circ ev_C \circ (e_C \otimes C) = ev_D \circ (e_B \otimes D) \circ (I \otimes f)$, and this holds trivially using Equation (11).

By definition of $\tilde{m}$ (see Eq. (1)) $\mu_S : (S, m) \to (S + I, \tilde{m})$ is a morphism of semigroups. Thus, by $(\tilde{\mathsf{S}}, \tilde{\mathsf{S}} \xrightarrow{\gamma^*}_1 \mathsf{End}(C)) \mapsto (\mathsf{S}, \mathsf{S} \xrightarrow{\gamma^* \circ \mu_S} \mathsf{End}(C))$, one assigns to every ${}_{\tilde{\mathsf{S}}}\mathbf{Lact}$-object an ${}_{\mathsf{S}}\mathbf{Lact}$-object. This construction obviously defines a functor Φ with $\Phi(f) = f$ for every morphism in ${}_{\tilde{\mathsf{S}}}\mathbf{Lact}$.

Now $\Psi \circ \Phi$ acts on monoid morphisms $\tilde{\mathsf{S}} \xrightarrow{\gamma^*} {}_1\mathsf{End}(C)$as

$$S + I \xrightarrow{\gamma^*} [C, C] \ \mapsto \ S \xrightarrow{\gamma^* \circ \mu_S} [C, C] \ \mapsto S + I \xrightarrow{\widetilde{\gamma^* \circ \mu_S}} [C, C];$$

by definition, $\widetilde{\gamma^* \circ \mu_S}$ is the unique monoid morphism $\mathsf{S} \to_1 \mathsf{End}(C)$ satisfying $(\widetilde{\gamma^* \circ \mu_S}) \circ \mu_S = \gamma^* \circ \mu_S$. Thus, $\Psi \circ \Phi(\gamma^*) = \gamma^*$.

Since $\Phi \circ \Psi$ acts on semigroup morphisms $\mathsf{S} \xrightarrow{\gamma^*} \mathsf{End}(C)$ as

$$S \xrightarrow{\gamma^*} [C, C] \ \mapsto \ S + I \xrightarrow{\tilde{\gamma}} [C, C] \ \mapsto \ S \xrightarrow{\tilde{\gamma} \circ \mu_S} [C, C],$$

one concludes $\Phi \circ \Psi(\gamma^*) = \gamma^*$ by definition of $\tilde{\gamma}$.

Since both functors, Φ and Ψ, are concrete functors over **C** by definition, one concludes that they are mutually inverse. □

It is clear by the examples discussed in Section 4 in which situations this result can be applied (e.g., **Ab**, $\mathbf{Mod}_R$, $\mathbf{Ban}_1$). Note that, in order to apply a nonsymmetric or dualized version (e.g., for actions of *R*-rings or coalgebras), one cannot apply Proposition 22 directly but has to resort to the argument preceding Eq. (10).

5. CONCLUSIONS

Generalizing the familiar adjunction of a unity element to a semigroup and a ring, respectively, we have constructed a left adjoint "unitarization functor" to the forgetful functor from the category $\mathbf{Mon}\mathbb{C}$ of monoids in a monoidal category $\mathbb{C}$ into the category $\mathbf{Sgr}\mathbb{C}$ of semigroups, using only mild natural conditions on $\mathbb{C}$. This not only allows for wide range of examples from various areas of algebra as shown in Section 4, but—due to its categorical character—for applications in the area of coalgebra as well, yielding here a "counitarization functor," right adjoint to the forgetful functor from comonoids to co-semigroups. A nontrivial compatibility of these functors with dualization functors (like the dual algebra functor) has been established.

The question, to what extent this compatibility also applies to Sweedler's dual ring functors (see [41]) will be dealt with in a paper in preparation. As a further open problem we ask whether there is a 2-dimensional generalization of our construction provided.

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REFERENCES

[1] Abuhlail, J. (2013). Semiunital semimonoidal categories (applications to semirings and semicorings). *Theory and Applications of Categories* 28:123–149.

[2] Ageron, P. (1993). Logic and category theory without the axiom of identity. *Rapports de Recherche du GRAL*, 1993–5 University of Caen.

[3] Aguiar, M., Mahajan, S. (2010). *Monoidal Functors, Species and Hopf Algebras*. CRM Monograph Series 29. Providence, RI: American Mathematical Society.

[4] Ambrose, W. (1945). Structure theorems for a special class of Banach algebras. *Trans. Amer. Math. Soc.* 57:364–386.

[5] Amitsur, S. A. (1968). Rings with involution. *Israel Journal of Mathematics* 6:99–106.

[6] Bourbaki, N. (2007). *Éléments de Mathématique–Algèbre (Chap. 1 à 3)*. Springer.

[7] Bergman, G. M., Hausknecht, A. O. (1996). *Cogroups and co-rings in categories of associative rings*. Memoirs of the American Mathematical Society 45.

[8] Borceux, F. (1994). *Handbook of categorical algebra 1*, Encyclopedia of Mathematics and its Applications 50, Cambridge: Cambridge University Press.

[9] Borceux, F. (1994). *Handbook of categorical algebra 2*, Encyclopedia of Mathematics and its Applications 51, Cambridge University Press.

[10] Brzezinski, T., Wisbauer, R. (2003). *Corings and Comodules*. Cambridge: Cambridge University Press.

[11] Cohn, P.M. (1995). *Skew fields - Theory of general division rings, Encyclopedia of mathematics and its applications* Vol. 57, Cambridge: Cambridge University Press.

[12] Dăscălescu, S., Năstăsescu, C., Raianu, Ş. (2001). *Hopf algebras - An introduction*, Pure and Applied Mathematics 235, Marcel Dekker, Inc.

[13] Deligne, P., Morgan, J.W., (1999). Notes on Supersymmetry (following Joseph Bernstein), *Quantum Fields and Strings: A Course for Mathematicians 1*, American Mathematical Society (1999), 41–97.

[14] Demazure, M., Gabriel, P. (1980). *Introduction to algebraic geometry and algebraic groups*, Amsterdam: North-Holland Mathematics Studies 39.

[15] Dixmier, J. (1977). *$C^\star$-Algebras*, Amsterdam - New York - Oxford: North-Holland.

[16] Dunford, N., Schwartz, J. T. (1963) *Linear operators. Part II: Spectral theory. Self adjoint operators in Hilbert space*, Pure and Applied Mathematics 8, Interscience Publishers.

[17] Helemskii, A. Ya., (2004). *Lectures and exercices on functional analysis*, Translations of Mathematical Monographs 233, American Mathematical Society.

[18] Herstein, I. N. (1976). *Rings with involution*, Chicago: Chicago Lectures in Mathematics.

[19] Heunen, C. (2013). On the functor ℓ^2. In: B. Coecke, L. Ong and P. Panangaden (Eds.): Abramsky Festschrift. *Lecture Notes in Computer Science* 7860:107–121.

[20] Hewitt, E., Ross, K. A. (1997). *Abstract harmonic analysis II*, Grundlehren der mathematischen Wissenschaften 152, Berlin: Springer-Verlag.

[21] Hochschild, G. P., (1971). *Introduction to affine algebraic groups*, San Francisco: Holden-Day.

[22] Jacobson, N. (1980). *Basic Algebra I*. San Francisco: W. H. Freeman and Company.

[23] Kadison R. V., Ringrose, J. R. (1983). *Fundamentals of the theory of operator algebras. Volume 1: Elementary theory*, Pure and Applied Mathematics, San Diego: Academic Press Inc.

[24] Kaplansky, I., (1957). *An introduction to differential algebra*, Publications de l'Institut Mathématiques de l'Université de Nancago V, Paris: Hermann.
[25] Linton, F. E. J. (1966). Autonomous equational categories, *J. Math. Mech.* 15:637–642.
[26] Mac Lane, S. (1963). *Homology*. Berlin-Göttingen-Heidelberg: Springer.
[27] Mac Lane, S. (1998). *Categories for the Working Mathematician*, 2nd ed., New York: Springer.
[28] Mitchell, B. (1983). Low dimensional group cohomology as monoidal structures. *American Journal of Mathematics* 105:1049–1066.
[29] Năstăsescu, C., Van Oystaeyen, F., (1979). *Graded and filtered rings and modules*, Lecture Notes in Mathematics 758, Springer.
[30] Pareigis, B. (1977). Non-additive ring and module theory I. General theory of monoids. *Publ. Math. Debrecen* 24:189–205.
[31] Penon, J. (1977). Sur les quasi-topos. *Cahiers de topologie et géométrie différentielle catégoriques* 18:181–218.
[32] Porst, H.–E. (2008). On categories of monoids, comonoids, and bimonoids. *Quaest. Math.* 31:127–139.
[33] Porst, H.–E. (2008). Dual adjunctions between algebras and coalgebras. *Arab. J. Sci. Eng.* 33 - 2 C:407–411.
[34] Porst, H.–E. (2012). Free Internal Groups. *Applied Categorical Structures* 20:31–42.
[35] Radford, D. E. (2011). *Hopf algebras*, K & E series on knots and everything 29, World Scientific.
[36] Rickart, C. E. (1974). *General Theory of Banach Algebras*. Huntington.
[37] Rieffel, M. A. (1972). Unitary representations induced from compact subgroups. *Studia Math* 42:145–175.
[38] Sakarovitch, J. (2009). Rational and recognisable power series, Chap. IV in Handbook of Weighted Automata, M. Droste, W. Kuich and H. Vogler, Eds., Springer, 105–174.
[39] Schatten, R. (1950). *A Theory of Cross-Spaces*, Annals of Mathematical Studies 26, Princeton University Press.
[40] Street, R. (2007). *Quantum Groups*. Cambridge: Cambridge University Press.
[41] Sweedler, M. (1975). The predual theorem to the Jacobson-Bourbaki theorem. *Trans. Amer. Math. Soc.* 213:391–406.
[42] Wyler, O. (1991). *Lecture Notes on Topoi and Quasitopoi*. World Scientific Publishing.