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Differential (Lie) algebras from a functorial point of view

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ABSTRACT

It is well-known that any associative algebra becomes a Lie algebra under the commutator bracket. This relation is actually functorial, and this functor, as any algebraic functor, is known to admit a left adjoint, namely the universal enveloping algebra of a Lie algebra. This correspondence may be lifted to the setting of differential (Lie) algebras. In this contribution it is shown that, also in the differential context, there is another, similar, but somewhat different, correspondence. Indeed any commutative differential algebra becomes a Lie algebra under the Wronskian bracket $W(a, b) = ab' - a'b$. It is proved that this correspondence again is functorial, and that it admits a left adjoint, namely the differential enveloping (commutative) algebra of a Lie algebra. Other standard functorial constructions, such as the tensor and symmetric algebras, are studied for algebras with a given derivation.

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1. Introduction and motivations

Given a commutative (associative) ring R with a unit, a *Lie R-algebra* (or just a *Lie algebra*) is a pair $(g, [-,-])$ where g is a R-module and $[-,-]: g \times g \to g$ is a *Lie bracket* (or just a *bracket*), i.e., a R-bilinear map which is

- *alternating*: this means that $[x, x] = 0$ for every $x \in g$,
- and satisfies the *Jacobi identity*: for every $x, y, z \in g$,

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0.$$

A Lie algebra is said to be *commutative* when its bracket is the zero bracket, i.e., $[x, y] = 0$ for every x, y. Thus it is essentially only a R-module.

Lie algebras are very common in the context of associative algebras since any (say unital) associative R-algebra $(A, *)$ may be turned into a Lie algebra when it is equipped with the so-called *commutator bracket*

$$[x, y] = x * y - y * x,$$

$x, y \in A$. The Lie algebra $(A, [-,-])$ is then referred to as the *underlying Lie algebra* of $(A, *)$. Because any algebra homomorphism induces a homomorphism of Lie algebras between the underlying Lie algebras, this correspondence between (unital) associative algebras and Lie algebras is actually functorial. Furthermore this functor is well-known to admit a left adjoint, namely the *universal enveloping algebra* $\mathsf{U}(g, [-,-])$ of a Lie algebra $(g, [-,-])$. It is given by $\mathsf{U}(g, [-,-]) = \mathsf{T}(g)/I$ where I is the two-sided ideal of the tensor R-algebra $\mathsf{T}(g)$ on the R-module g generated by $x \otimes y - y \otimes x - [x, y]$, $x, y \in g$ (for more details see e.g. [6]).

This algebra is *universal* among all (unital) algebras with a Lie map from $(g, [-,-])$ to its underlying Lie algebra. By this is meant the following. Let j_g be the composite R-linear map $g \hookrightarrow \mathsf{T}(g) \xrightarrow{\pi} \mathsf{U}(g, [-,-])$, where the first arrow is the canonical inclusion from the R-module g into its tensor algebra, and π is the canonical projection. The linear map j_g happens to be a homomorphism of Lie algebras from $(g, [-,-])$ to the underlying Lie algebra of $\mathsf{U}(g, [-,-])$. Now, given any other associative algebra $(B, *)$ with a unit and a homomorphism $\phi: (g, [-,-]) \to (B, [-,-])$ of Lie algebras, then ϕ uniquely factors through j_g, i.e., there exists a unique homomorphism of algebras $\hat{\phi}: (A, *) \to (B, *)$ such that the following diagram commutes (in the category of Lie algebras).

$$\begin{array}{ccc} (g, [-,-]) & \xrightarrow{\phi} & B \\ {\scriptstyle j_g}\downarrow & \nearrow_{\hat{\phi}} & \\ \mathsf{U}(g, [-,-]) & & \end{array} \tag{1}$$

This relation makes possible the study of Lie algebras within the realm of associative algebras. In this situation one may ask for the conditions under which a Lie algebra embeds into (the underlying Lie algebra of) its universal enveloping algebra; more precisely when is the canonical map j_g one-to-one? An answer, known as the *Poincaré–Birkhoff–Witt Theorem* (see again [6]), is as follows.

Theorem 1 *(Poincaré–Birkhoff–Witt Theorem). (See [23,4,30].) If the underlying R-module g of a Lie algebra $\mathfrak{g}$ is free, then j_g is one-to-one.*

Actually in its more general form this theorem tells us more: it provides a R-basis of the universal enveloping algebra $\mathsf{U}(g, [-,-])$, obtained from a R-basis of the module g. Other more elaborate results exist (e.g. [10]).

There are obvious counterparts of associative and Lie algebras in the differential setting, namely the same structures with an additional derivation. Hence one may wonder whether there is a way to extend also the notion of universal enveloping algebra in the differential setting.

The answer is affirmative as it is explained in this paper. And there are even (at least) two different ways to proceed. In the first one, one lifts the constructions and results from the classical situation to the differential setting. This is possible because any derivation on an algebra is also a derivation for its commutator bracket, and, moreover, the usual universal enveloping algebra on a Lie algebra $\mathfrak{g}$ which admits a derivation may be equipped with a derivation that extends the derivation of $\mathfrak{g}$. It follows at once that the Poincaré–Birkhoff–Witt Theorem remains unchanged. The second way to treat the relations between differential Lie algebras and differential algebras, also discussed hereafter, is quite different since based on the “Wronskian bracket”, instead of the commutator bracket. Thus it depends on the derivation in an essential way. A sketch of this approach is presented now for the reader’s convenience and to motivate subsequent developments.

Let us give a differential commutative algebra $((A, *), d)$. Its *Wronskian bracket* is defined as

$$W(x, y) = x * d(y) - d(x) * y, \quad x, y \in A$$

and it turns A into a differential Lie algebra. Similarly to the commutator bracket, this defines a functor from differential algebras to differential Lie algebras. Therefore one can ask a few questions:

1. Does this functor admit a left adjoint? In other terms, is there a corresponding universal enveloping differential commutative algebra? The answer is positive and follows from a general theorem from universal algebra.
2. Under which conditions on the Lie algebra and on the base ring is the canonical map from a differential Lie algebra to its differential enveloping algebra one-to-one? This question is still open.

Only the first of the above two questions is addressed in this paper. Nevertheless I provide an example and a counter-example illustrating the fact that the second question seems to be more difficult than the same problem in a non-differential context, which already has an answer given by the Poincaré–Birkhoff–Witt Theorem.

The presentation of the two approaches for the universal enveloping algebra construction on a differential Lie algebra is not the unique subject of this contribution. I also provide a uniform and general treatment of functorial constructions (e.g., tensor or symmetric algebras) between categories of differential algebras of several kinds, that I call "universal differential algebra" in order to focus on the use of techniques from universal algebra. Hence it is also both a continuation of, and a complement to, the article [25]. Furthermore, I point out an application to the fragment, developed here, of universal differential algebra to Rota–Baxter (Lie) algebras, whose main merit is to show the usefulness of the contribution in another branch than differential algebra.

A large part of the content of this paper has been presented during a talk, with the same title as this contribution, given in the special session "Algebraic and Algorithmic aspects of Differential and Integral Operators" AADIOS 2014 at the conference ACA 2014 held at New York in July 2014.

Finally I mention a remark by the referee of the paper. There already exists a notion of differential algebraic Lie algebras [8]. These objects are rather different from mine because of the absence of a derivation, in general, in the Lie algebras. Differential algebraic Lie algebras may be better seen as the differential counterpart of affine algebraic group schemes, thus merely as the solution spaces of some systems of differential equations. Moreover, the referee pointed out that the Wronskian bracket already appeared for instance in [26] as the antisymmetrization of the second-degree terms of a one-dimensional formal differential group known as the "substitution group". The connections between both notions of differential Lie algebras are not studied in this contribution.

Organization of the paper

In Section 2 and in Section 3 are recalled some notions from category theory and from universal algebra which are frequently used hereafter. Section 4 contains an explicit description of a left adjoint of an algebraic functor (between equational varieties of algebras or concrete categories concretely isomorphic to varieties) which makes possible to treat many functorial constructions in a uniform way, and which is applied in Section 5, where are also introduced some basic concepts of "universal differential algebra", whose objective consists in dealing with equational varieties of differential algebras of several kinds. In particular, the tensor, symmetric and group algebra constructions are detailed in the differential algebra setting. The two approaches for the universal enveloping algebra construction for differential Lie algebras – namely by lifting the usual construction and by using the "Wronskian" bracket instead of the commutator bracket – are developed in Section 6. A similar construction for Rota–Baxter (Lie) algebras is given in Section 7.

2. Basic notions from category theory

In this section are summarized some notions from category theory that will be needed for the rest of this contribution and that, for the essential part, concern left adjoint functors (free constructions) and concrete categories. Our main reference is [21].

For a category $\mathbf{C}$, it is always assumed – in this contribution – that the class of all morphisms from an object c to another one c' forms a set, which is denoted by $\mathbf{C}(c, c')$. (Some authors refer to as *locally small* for this kind of categories.) By a $\mathbf{C}$-*object* is meant an "object of $\mathbf{C}$", and similarly for morphisms (the term "map" will also be used a few times for "morphism").

Let $\mathbf{C}$ be a category, and let E be any sub-class of objects of $\mathbf{C}$. This characterizes a particular category $\mathbf{C}E$, the *full sub-category* spanned by E: for each $e, e' \in E$, one has $\mathbf{C}E(e, e') = \mathbf{C}(e, e')$, and the composition is that of $\mathbf{C}$. More generally a *full sub-category* of a category $\mathbf{C}$ is a category of the form $\mathbf{C}E$ for some E.

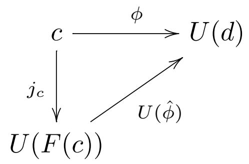
The *identity functor* on a category $\mathbf{C}$ is the functor which is the identity map both on objects and on arrows. A functor $F: \mathbf{C}_1 \to \mathbf{C}_2$ is called an *isomorphism*, and the two categories are said to be *isomorphic*, when there exists a functor $G: \mathbf{C}_2 \to \mathbf{C}_1$ such that $F \circ G$ is the identity functor on $\mathbf{C}_2$ and $G \circ F$ is the identity functor on $\mathbf{C}_1$. One also says that $\mathbf{C}_1$ may be *identified* with $\mathbf{C}_2$.

Let $\mathbf{C}$ and $\mathbf{D}$ be two categories, and let $U: \mathbf{D} \to \mathbf{C}$ be a functor. A *left adjoint* of U is a functor $F: \mathbf{C} \to \mathbf{D}$ such that there exists a set-theoretic bijection $\tau_{c,d}: \mathbf{C}(c, U(d)) \simeq \mathbf{D}(F(c), d)$, natural in c and in d (where c is a "variable" object of $\mathbf{C}$ while d is a "variable" object of $\mathbf{D}$), i.e., for every $h \in \mathbf{C}(c', c)$ and $f \in \mathbf{D}(d, d')$, the following diagram commutes (in the category $\mathbf{Set}$ of sets).

$$\begin{array}{ccc} \mathbf{C}(c, U(d)) & \xrightarrow{\tau_{c,d}} & \mathbf{D}(F(c), d) \\ {\scriptstyle g \mapsto U(f)\circ g\circ h}\downarrow & & \downarrow{\scriptstyle g \mapsto f\circ g\circ F(h)} \\ \mathbf{C}(c', U(d')) & \xrightarrow[\tau_{c',d'}]{} & \mathbf{D}(F(c'), d') \end{array} \tag{2}$$

Remark 2. The notion of a *right adjoint* is obtained by dualizing (i.e., reversing the direction of all arrows) the definition of a left adjoint: F is a left adjoint of U, if, and only if, U is a right adjoint of F.

For each $\mathbf{C}$-object c, $F(c)$ may be referred to as the *free* $\mathbf{D}$-*object generated by* c and the canonical $\mathbf{C}$-morphism $j_c := \tau^{-1}_{c,F(c)}(id_{F(c)}) \in \mathbf{C}(c, U(F(c)))$ is sometimes called the *insertion of generators* by analogy with the free object constructions in universal algebra, see Section 3 below (even if it is not, in general, a monomorphism) or also the *unit of the adjunction*. Also, if d is an object of $\mathbf{D}$, $U(d)$ may be referred to as the *underlying* $\mathbf{C}$-*object* of d. Unfolding the previous definition the following *universal property* is satisfied by $F(c)$: for every $\mathbf{C}$-morphism $\phi: c \to U(d)$, where d is any $\mathbf{D}$-object, there is a unique **D**-morphism $\hat{\phi}: F(c) \to d$ such that the following triangle commutes in **C**.

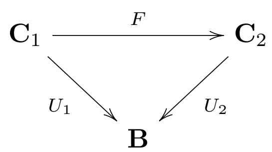


(3)

From this universal property, the action of F on arrows is recovered as $F(f) = (\widehat{j_{c'} \circ f}) \in \mathbf{D}(F(c), F(c'))$ for $f \in \mathbf{C}(c, c')$ (hence $U(F(f)) \circ j_c = j_{c'} \circ f$). In conclusion, a universal property satisfied by a functor and its values on objects suffice to determine its values on arrows.

Example 3. Let R be a commutative ring with a unit. Let ${}_R\mathbf{Mod}$ be the category of all R-modules, and let $U: {}_R\mathbf{Mod} \to \mathbf{Set}$ be the functor that forgets the modules structure and so assigns its carrier set to a module. The well-known free module construction provides a left adjoint of U (see e.g. [7]).

A pair $(\mathbf{C}, U)$, where $\mathbf{C}$ is a category and U is a functor from $\mathbf{C}$ to some category $\mathbf{B}$, is called a *category concrete over* $\mathbf{B}$ (or simply a *concrete category* when $\mathbf{B} = \mathbf{Set}$). Given two concrete categories $(\mathbf{C}_i, U_i)$, $i = 1, 2$, over $\mathbf{B}$, a functor $F: \mathbf{C}_1 \to \mathbf{C}_2$ is said to be a *concrete functor* if the following triangle commutes.



(4)

In case F is an isomorphism, one says that $(\mathbf{C}_1, U_1)$ and $(\mathbf{C}_2, U_2)$ are *concretely isomorphic*, and F is said to be a *concrete isomorphism*. (One observes that the inverse G of the concrete isomorphism F is also concrete since $U_2 \circ F = U_1$ implies $U_2 = U_2 \circ F \circ G = U_1 \circ G$.)

Finally, a functor $F: \mathbf{C} \to \mathbf{D}$ is said to be *faithful* if for all $\mathbf{C}$-objects c, c', given two $\mathbf{C}$-morphisms $f, g \in \mathbf{C}(c, c')$, then $F(f) = F(g)$ implies that $f = g$. The functor $F: \mathbf{C} \to \mathbf{D}$ is said to be *full* if for all $\mathbf{C}$-objects c, c', and for each $g \in \mathbf{D}(F(c), F(c'))$, there exists a $\mathbf{C}$-morphism $f \in \mathbf{C}(c, c')$ such that $F(f) = g$.

Example 4. The category of monoids is concrete (over **Set**) using the obvious forgetful functor, which is faithful (as it is the case for any variety of algebras; see Section 3), that consists in forgetting the multiplication and the identity element of a monoid.

3. A glance at universal algebra

Concrete categories play a fundamental role in universal algebra because each variety of algebras may be seen as a concrete category (over **Set**) with its obvious forgetful functor, which sends an algebra to its carrier set.

Let me briefly recall some concepts from universal algebra first informally and then more formally (see [11] for more details). Roughly speaking an "algebra" is a set A equipped with a number, finite or not, of distinguished "basic" operations on it, i.e., mappings from A^n to A for various n, called "n-ary" operations. For instance a monoid $(M, *, e)$ may be described as a set M with two operations, its multiplication $*: M^2 \to M$, and its identity element (seen as an operation from M^0 to M).

Given two "similar" algebras A, B, i.e., algebras with, for each n, the same number of n-ary operations, a homomorphism of algebras is defined to be a set-theoretic map $\phi: A \to B$ that "commutes" with the operations: if f is an n-ary operation of A, and if g is the corresponding operation on B, then for each $a_1, \ldots, a_n \in A$, $\phi(f(a_1, \ldots, a_n)) = g(\phi(a_1), \ldots, \phi(a_n))$. For instance homomorphisms of monoids are required to commute with the multiplications, $\phi(a_1 * a_2) = \phi(a_1) * \phi(a_2)$ and with the identity elements, $\phi(e) = e$.

But operations are not sufficient to do algebra since one also needs equations. A set of basic operations generates, under "superposition" new operations, called "derived operations". For instance the commutator $[-,-]: A^2 \to A$, defined by $(a_1, a_2) \mapsto [a_1, a_2] = a_1a_2 - a_2a_1$, is a derived operation of any (usual) associative algebra A. In universal algebra, an equation thus is a pair of derived operations. For instance, the associativity relation in an algebra $(M, *)$, with a unique binary operation $*$, may be described as the equation (g, d), where $g, d: M^3 \to M$ are defined by $g(a_1, a_2, a_3) = (a_1 * a_2) * a_3$, and $d(a_1, a_2, a_3) = a_1 * (a_2 * a_3)$. Such an equation holds in an algebra whenever the two members of the equation define the same map. Hence, in any semigroup, the associativity relation holds.

The class of algebras characterized by a certain set of basic operations and the members of which satisfy some given (finite or not) set of equations forms an "equational variety". From a category-theoretic point of view, these varieties have nice properties. For instance, they admit all (small) categorical products and coproducts. A more important result, at least for this contribution, to be stressed is that any carrier set preserving functors between varieties have a left adjoint. This is an essential property recalled in detail in this section and used in this contribution.

I now continue with a more formal approach to universal algebra. Of course a drawback of such a description is perhaps a cumbersome symbolism but I believe that such a rigorous treatment is required to provide formal proofs.

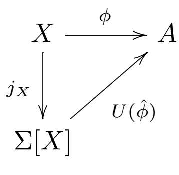
A (*one-sorted* or *homogeneous* and *finitary*) *signature* is a collection of sets $(\Sigma(n))_{n\in\mathbb{N}}$. (Observe that $\Sigma(n)$ may be empty for some n and these sets are not assumed to be pairwise distinct.) A member f of $\Sigma(n)$ is referred to as a *function symbol of arity n*. A Σ-*algebra* is a pair (A, F) where A is a set and $F = (F_n)_{n\in\mathbb{N}}$ is a collection of maps

$F_n\colon \Sigma(n) \to A^{A^n}$ for each n; so F assigns an operation of n variables on A to a function symbol of arity n. In practice the subscript n on F_n, and even F itself, are omitted.

Given two Σ-algebras (A, F) and (B, G), a *homomorphism of Σ-algebras* $\phi\colon (A, F) \to (B, G)$ is a set-theoretic map $\phi\colon A \to B$ such that $\phi(F_n(f)(a_1, \ldots, a_n)) = G_n(f)(\phi(a_1), \ldots, \phi(a_n))$ for each $n \in \mathbb{N}$ and each $f \in \Sigma(n)$, $a_1, \ldots, a_n \in A$. With the usual composition of maps, one gets a category of all Σ-algebras. One observes that this category is a concrete category ($U\colon (A, F) \mapsto A$ defines, at the level of objects, a functor from Σ-algebras to sets). This functor admits a left adjoint, known as the *free Σ-algebra construction*: for a set X, $\Sigma[X]$ then denotes the *free Σ-algebra on* X, also called the *term algebra* because its members are usually referred to as *terms*.

Remark 5. With a notation such as "$\Sigma[X]$", the Σ-algebra structure is not specified, so that the algebra and, by this abuse of notation, its carrier set are identified, i.e., $\Sigma[X] = U(\Sigma[X])$. This is legitimate by the following. The members of $\Sigma[X]$, seen as a sub-set of the free semigroup on $X \sqcup \{(\,\} \sqcup \{\,)\} \sqcup \bigsqcup_{n\in\mathbb{N}} \Sigma(n)$, may be defined recursively as follows: $j_X(x) \in \Sigma[X]$ for every $x \in X$ (where j_X is the canonical injection from $X \sqcup \{(\,\} \sqcup \{\,)\} \sqcup \bigsqcup_{n\in\mathbb{N}} \Sigma(n)$ to the free semigroup), and for each $f \in \Sigma(n)$, $t_1, \ldots, t_n \in \Sigma[X]$ implies that $f(t_1, \ldots, t_n) \in \Sigma[X]$. This representation, used in [1], differs from that in [11] by the use of parentheses. Within this representation, the Σ-algebra structure on $\Sigma[X]$ is defined by $F_n(f)(t_1, \ldots, t_n) = f(t_1, \ldots, t_n) \in \Sigma[X]$ for each $f \in \Sigma(n)$ (observe that in this equality the parentheses that occur in the right hand-side are just symbols as is, for instance, f, since the right hand-side is itself a word obtained by concatenation), $t_1, \ldots, t_n \in \Sigma[X]$. This is because it is so transparent that F is usually omitted.

The free algebra thus satisfies the following (see Section 2): given any Σ-algebra (A, F), and any set-theoretic map $\phi\colon X \to A$, there is a unique homomorphism of Σ-algebras $\hat{\phi}\colon \Sigma[X] \to (A, F)$ such that the following diagram commutes, where as in Section 2, $j_X\colon X \to \Sigma[X]$ denotes the insertion of generators (which, in this case, is injective).



(5)

Definition 6. By a *congruence* on a Σ-algebra (A, F) is meant an equivalence relation on the set A, closed under F, i.e., if $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are equivalent, then for each $f \in \Sigma(n)$, $F_n(f)(a_1, \ldots, a_n)$ and $F_n(f)(b_1, \ldots, b_n)$ also are equivalent. Moreover a congruence is *fully invariant* if it is closed under endomorphisms, i.e., if $\phi\colon (A, F) \to (A, F)$ is a homomorphism, hence an "endomorphism", and a, b are equivalent, then $\phi(a)$, $\phi(b)$ are also equivalent. Any sub-set E of a Σ-algebra (A, F) *generates* a congruence (respectively, a fully invariant congruence) namely the intersection of all (fully invariant) congruences of (A, F) that contain E.

A *law* or *equation* is a pair $(u, v) \in \Sigma[\mathbb{N}]^2$. Given a Σ-algebra (A, F), an equation (u, v) is said to be *satisfied* or *valid* in (A, F), or that (A, F) *satisfies* (u, v), whenever for every map $\phi: \mathbb{N} \to A$, $\hat{\phi}(u) = \hat{\phi}(v)$. In practice a law (u, v) is denoted by an equality $u = v$. A(n *equational*) *variety* of Σ-algebras is defined as a class of Σ-algebras that satisfy some fixed set E of laws (for instance $E = \emptyset$ for the variety of all Σ-algebras).

Several sets of laws may define the same variety, and any such a set is referred to as a set of *defining equations* for the variety. Nevertheless any of such a set of defining equations for a given variety generates the same fully invariant congruence on $\Sigma[\mathbb{N}]$, which is the least fully invariant congruence on $\Sigma[\mathbb{N}]$ that contains the set of defining equations (see Definition 6), and actually, there is a one–one correspondence between varieties of Σ-algebras and fully invariant congruences on $\Sigma[\mathbb{N}]$ (see [11, Chap. IV] for the details). The celebrated *Birkhoff's HSP Theorem* [11] provides a "semantic" characterization of varieties: a class of Σ-algebras is an equational variety if, and only if, it is closed under homomorphic images (H), sub-algebras (S) and products (P).

Remark 7. In the above paragraph, the set $\mathbb{N}$ is just a canonical choice and may be replaced by any infinite countable set such as for instance $\{x_i : i \in \mathbb{N}\}$, with $x_i \neq x_j$ for all $i \neq j$. The only important feature is the possibility to introduce as many new variables as desired.

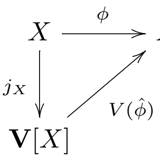
In what follows a variety $\mathbf{V}$ is always identified with the full sub-category of all Σ-algebras spanned by the members of the variety. Within this identification, a variety is even a concrete category (with the restriction V of the forgetful functor U of the category of all Σ-algebras). It is easily checked that the forgetful functor V of a variety $\mathbf{V}$ is a faithful functor. I thus often identify a homomorphism $\phi: (A, F) \to (B, G)$ of Σ-algebras in the variety $\mathbf{V}$ with its underlying set-theoretic map $V(\phi): A \to B$.

Remark 8. The quotient of a Σ-algebra, in some variety $\mathbf{V}$, by a congruence inherits a structure of a Σ-algebra in the same variety, and the canonical projection becomes a homomorphism of Σ-algebras. Moreover the kernel $\ker\phi$ of a homomorphism ϕ of algebras (i.e., the equalizer $\{(x, y): \phi(x) = \phi(y)\}$ of ϕ with itself) is a congruence, and all congruences arise as kernels.

A concrete functor between varieties (not necessarily on the same signature) is called an *algebraic functor* (see [20]). Hence an algebraic functor is just a functor between varieties of algebras (possibly over different signatures) that makes commute a triangle as in Diagram (4). One of the great achievement of universal algebra is the following fundamental result.

Theorem 9. *(See [3, Corollary 8.17, p. 28].) Each algebraic functor admits a left adjoint.*

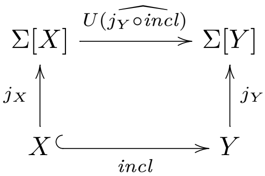
Remark 10. The existence of a free Σ-algebra follows at once from Theorem 9 since the forgetful functor U is itself an algebraic functor (the category of sets may be considered as the variety of Σ-algebras, where Σ is the empty signature, i.e., $\Sigma(n) = \emptyset$ for each $n \in \mathbb{N}$). Similarly, for every variety $\mathbf{V}$ of Σ-algebras, due to the existence of a left adjoint of the forgetful functor V of the variety, a free construction is available. This reads as follows. Let X be any set. One denotes by $\mathbf{V}[X]$ the free Σ-algebra in the variety $\mathbf{V}$ generated by X (the carrier set and the algebra structure are once again identified). Let (A, F) be an object of $\mathbf{V}$, and let $\phi \colon X \to A$ be any set-theoretic map. Then, there exists a unique homomorphism of Σ-algebras $\hat{\phi} \colon \mathbf{V}[X] \to (A, F)$ such that the following diagram commutes (in the category of sets), where j_X is the insertion of generators.



(6)

A last observation: j_X is one-to-one if, and only if, $\mathbf{V}$ is a non-trivial variety (a *trivial variety* is a variety without algebras with more than one element). For instance a basis for a free group may be identified with a sub-set within the free group.

Remark 11. If $X \subseteq Y$ are sets, then $\Sigma[X]$ may be canonically identified with a sub-algebra of $\Sigma[Y]$ using the homomorphism $(\widehat{j_Y \circ incl}) \colon \Sigma[X] \to \Sigma[Y]$ given by the following commutative diagram (with the same notations as in Diagram (5)), where $incl \colon X \hookrightarrow Y$ denotes the canonical inclusion.



(7)

More generally let $\phi \colon X \to Y$ be a one-to-one map, and let $r \colon Y \to X$ be a retraction of ϕ, i.e., $r \circ \phi = id_X$. One has $\widehat{j_X \circ r} \circ \widehat{j_Y \circ \phi} = id_{\Sigma[X]}$, hence $\widehat{j_Y \circ \phi}$ is also one-to-one, because $\widehat{j_X \circ r} \circ \widehat{j_Y \circ \phi} \circ j_X = \widehat{j_X \circ r} \circ j_Y \circ \phi = j_X \circ r \circ \phi = j_X$, $id_{\Sigma[X]} \circ j_X = j_X$ and the universal property (see Section 2) satisfied by $\Sigma[X]$.

A concrete category concretely isomorphic to a variety of algebras is not necessarily itself a variety (e.g., consider the category of monoids the members of which are invertible; it is a concrete category, a full sub-category of the variety of all monoids, and it is concretely isomorphic to the variety of groups, but it is not a variety since it is not closed under sub-algebras: for instance the sub-monoid $\mathbb{N}$ of $\mathbb{Z}$). According to Lemma 12 below, there is no danger to identify a concrete category with a concretely isomorphic variety (if any) as soon as the identification concerns free objects or a left adjoint of a forgetful functor. For other properties of varieties one should be more cautious.

Lemma 12. *Let* $\mathbf{V}$ *be a variety of say* Σ*-algebras and let* $(\mathbf{C}, U)$ *be a concrete category over* $\mathbf{Set}$. *Let us assume that* $(\mathbf{C}, U)$ *and* $(\mathbf{V}, V)$ *are concretely isomorphic. Then,* U *admits a left adjoint.*

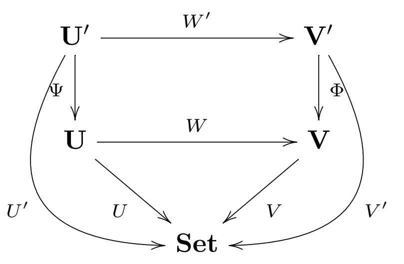
Proof. Let us consider a concrete isomorphism $\Phi\colon (\mathbf{C}, U) \to (\mathbf{V}, V)$. Of course, its inverse Φ^{-1} also is a concrete isomorphism. Let X be any set. Let c be a $\mathbf{C}$-object and let $\phi\colon X \to U(c) = V(\Phi(c))$ be a set-theoretic map. There is a unique homomorphism of Σ-algebras $\hat{\phi}\colon \mathbf{V}[X] \to \Phi(c)$ such that $V(\hat{\phi}) \circ j_X = \phi$, where $j_X\colon X \to V(\mathbf{V}[X]) = U(\Phi^{-1}(\mathbf{V}[X]))$ is the insertion of generators. Let us consider the $\mathbf{C}$-morphism $\Phi^{-1}(\hat{\phi})\colon \Phi^{-1}(\mathbf{V}[X]) \to c$. One has $U(\Phi^{-1}(\hat{\phi})) \circ j_X = V(\hat{\phi}) \circ j_X = \phi$. Let us assume that there exists another $\mathbf{C}$-morphism $\psi\colon \Phi^{-1}(\mathbf{V}[X]) \to c$ with $U(\psi) \circ j_X = \phi$. Then, $V(\Phi(\psi)) \circ j_X = \phi$, hence by uniqueness of $\hat{\phi}$ it follows that $\hat{\phi} = \Phi(\psi)$. □

Example 13. Let us again consider the category of monoids the members of which are invertible, which is concretely isomorphic to the variety of all groups. Then, the free object in the former category generated by a set X is, according to (the proof of) Lemma 12, the underlying monoid of the free group on X.

The following more general result than Lemma 12 is of a fundamental interest for the rest of this paper, and shows that there is no danger to replace – what will be done without any further ado – a variety by a concretely isomorphic category as soon as it concerns left adjoints of algebraic functors of the following particular kind. In particular one can talk about *algebraic functors* between concrete categories concretely isomorphic to varieties, and these functors also admit a left adjoint.

Lemma 14. *Let* $(\mathbf{U}, U), (\mathbf{V}, V)$ *be varieties of algebras, and let* $(\mathbf{U}', U'), (\mathbf{V}', V')$ *be concrete categories which are concretely isomorphic with respectively* $(\mathbf{U}, U)$ *and* $(\mathbf{V}, V)$ *by concrete isomorphisms* Ψ *and* Φ. *Let* $W\colon (\mathbf{U}, U) \to (\mathbf{V}, V)$ *be an algebraic functor. Then, the functor* $W' = \Phi^{-1} \circ W \circ \Psi\colon \mathbf{U}' \to \mathbf{V}'$ *is a concrete functor, and it also admits a left adjoint.*

Proof. The situation stated in this lemma is represented by the following commutative diagram of functors.



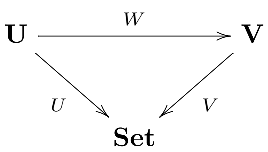
(8)

I claim that $\Psi^{-1} \circ L \circ \Phi \colon \mathbf{V}' \to \mathbf{U}'$ is a left adjoint of W', where $L \colon \mathbf{V} \to \mathbf{U}$ is a left adjoint of W (which exists because W is an algebraic functor). This follows from the obvious bijections, natural in the $\mathbf{V}'$-object v and in the $\mathbf{U}'$-object u.

$$\begin{aligned}
\mathbf{V}'(v, W'u) &= \mathbf{V}'(v, \Phi^{-1}(W(\Psi(u)))) \\
&\simeq \mathbf{V}(\Phi(v), W(\Psi(u))) \\
&\quad \text{(because any functor preserves isomorphisms)} \\
&\simeq \mathbf{U}(L(\Phi(v)), \Psi(u)) \\
&\quad \text{(since } L \text{ is a left adjoint of } W\text{)} \\
&\simeq \mathbf{U}'(\Psi^{-1}(L(\Phi(v))), u). \qquad \square
\end{aligned} \tag{9}$$

4. A general construction of a left adjoint of an algebraic functor

Let Σ, Ω be two finitary homogeneous signatures. Let $\mathbf{U}$ (respectively, $\mathbf{V}$) be a variety of Σ-algebras (respectively, Ω-algebras). Let us assume that there is an algebraic functor $W \colon \mathbf{U} \to \mathbf{V}$, i.e., the following diagram of functors commutes (where U and V are the obvious forgetful functors).



(10)

Thus the underlying sets of an object $\mathcal{B}$ of the domain category $\mathbf{U}$ and of the object $W(\mathcal{B})$ are the same. According to [29, Theorem 1.4], this is equivalent to the fact that $\mathbf{V}$ is *representable* in $\mathbf{U}$. By this is meant that for each n and for each $f \in \Omega(n)$, there exists a n-ary term $\bar{f} \in \Sigma[\mathbb{N}]$ (i.e., $\bar{f}$ is a term in $\Sigma[\{0, \ldots, n-1\}]$, seen as a sub-set of $\Sigma[\mathbb{N}]$; see Remark 11) such that if e is a defining equation of the variety $\mathbf{V}$, then the equation $\bar{e}$, obtained from e by replacing each occurrence of f by the corresponding $\bar{f}$, is a consequence of the defining equations of the variety $\mathbf{U}$ (i.e., $\bar{e}$ belongs to the fully invariant congruence of $\Sigma[\mathbb{N}]$ that corresponds to the variety $\mathbf{U}$). The collection of maps $(f \in \Omega(n) \mapsto \bar{f} \in \Sigma[\{0, \ldots, n-1\}])_n$ satisfying the above properties is called a *representation of* $\mathbf{V}$ *in* $\mathbf{U}$.

Remark 15. Let (A, F) be a Σ-algebra. Any term t in the free Σ-algebra $\Sigma[\{0, \ldots, n-1\}]$ defines a map $[t] \colon A^n \to A$ by $[t](a_1, \ldots, a_n) = \hat{\pi}_{a_1,\ldots,a_n}(t)$, where $\pi_{a_1,\ldots,a_n} \colon \{0, \ldots, n-1\} \to A$ is given by $\pi(i-1) = a_i$, $i = 1, \ldots, n$ (using the notation from Diagram (5)). The substitution of the occurrences of a function symbol f by $\bar{f}$ in a term $t \in \Sigma[\mathbb{N}]$, abovementioned to define an equation $\bar{e}$, is then possible because $\Sigma[\mathbb{N}]$ is equipped with a structure of an Ω-algebra given by $F_n(f) = [\bar{f}] \colon \Sigma[\mathbb{N}]^n \to \Sigma[\mathbb{N}]$ for each $f \in \Omega(n)$.

Example 16. The variety of all Lie algebras on a ring R is representable in the variety of all associative R-algebras. Indeed, the Lie bracket $[-,-]$, which is a basic operation in the signature of Lie algebras, corresponds to the derived operation $\overline{[-,-]}(x_0,x_1) = (x_0 * x_1) - (x_1 * x_0)$, the usual commutator bracket. From the Jacobi identity, given for instance has an equation $e = ([x_0,[x_1,x_2]], [[x_0,x_1],x_2] + [x_1,[x_0,x_2]])$, one gets the new equation $\bar{e}$ given by

$$\begin{aligned} &x_0 * ((x_1 * x_2) - (x_2 * x_1)) - ((x_1 * x_2) - (x_2 * x_1)) * x_0 \\ &\quad = ((x_0 * x_1) - (x_1 * x_0)) * x_2 - x_2 * ((x_0 * x_1) - (x_1 * x_0)) \\ &\qquad + x_1 * ((x_0 * x_2) - (x_2 * x_0)) - ((x_0 * x_2) - (x_2 * x_0)) * x_1 \end{aligned} \tag{11}$$

(after replacement in e of the Lie brackets by the commutator brackets). This equation $\bar{e}$ holds in any associative algebra since it is a consequence of the defining axioms. Similarly, the equation for alternativity $f = ([x_0,x_0],0)$ gives $\bar{f} = (x_0 * x_0 - x_0 * x_0, 0)$.

So let us get back to the situation where the Diagram (10) commutes. Given any Σ-algebra (B,G) in $\mathbf{U}$, then the Ω-algebra $W(B,G)$ in $\mathbf{V}$ is given as follows. First of all, its carrier set is B because $V(W(B,G)) = U(B,G) = B$. Now, for each $f \in \Omega(n)$ is defined $[\bar{f}]: B^n \to B$ as in Remark 15. This provides a map $G_n: \Omega(n) \to B^{B^n}$, for each n, that turns B into a Ω-algebra in $\mathbf{V}$ (each defining equation e of $\mathbf{V}$ holds in $W(B,G)$ since $\bar{e}$ is a consequence of the defining equations of $\mathbf{U}$).

Since W is an algebraic functor, it admits a left adjoint. Let us make explicit the description of such a left adjoint. Let (A,F) be an Ω-algebra in $\mathbf{V}$. Let $\mathbf{U}[V(A,F)] = \mathbf{U}[A]$ be the free Σ-algebra in the variety $\mathbf{U}$ generated by the set $V(A,F) = A$. Let $j_A = j_{V(A,F)}: V(A,F) \to U(\mathbf{U}[V(A,F)])$ be, as usual, the "insertion of generators". Let us consider the least congruence $\equiv$ on $\mathbf{U}[V(A,F)] = \mathbf{U}[A]$ generated by the pairs

$$(j_A(f(a_1,\dots,a_n)), [\bar{f}](j_A(a_1),\dots,j_A(a_n)))$$

for all n, $f \in \Omega(n)$, $a_1,\cdots,a_n \in A$, where $[\bar{f}]$ is as in Remark 15 (in particular, with $f \in \Sigma(0)$, then it reduces to $(j_A(f),[\bar{f}])$). Finally, let $\pi: \mathbf{U}[V(A,F)] \to \mathbf{U}[V(A,F)]/\equiv$ be the natural epimorphism, which is a homomorphism of Σ-algebras, where $\mathbf{U}[V(A,F)]/\equiv$ is given the quotient Σ-algebra structure $F_n(f)(\pi(t_1),\dots,\pi(t_n)) = \pi(f(t_1,\dots,t_n))$ for $f \in \Sigma(n)$ and $t_1,\dots,t_n \in \mathbf{U}[V(A,F)]$.

Lemma 17. *The composite* $i_{(A,F)} = A = V(A,F) \xrightarrow{j_A} U(\mathbf{U}[V(A,F)]) \xrightarrow{U(\pi)} U(\mathbf{U}[V(A,F)]/\equiv)$ *is actually a homomorphism, again denoted by* $i_{(A,F)}$ *(and thus* $V(i_{(A,F)}) = U(\pi) \circ j_A$*), of* Ω*-algebras from* (A,F) *to* $W(\mathbf{U}[V(A,F)]/\equiv)$.

Proof. It suffices to check that $i_{(A,F)}(f(a_1,\dots,a_n)) = [\bar{f}](i_{(A,F)}(a_1),\dots,i_{(A,F)}(a_n))$ for each $f \in \Omega(n)$, $a_1,\dots,a_n \in A$. One has

$$\begin{aligned}\pi(j_A(f(a_1,\ldots,a_n))) &= \pi([\bar{f}](j_A(a_1),\ldots,j_A(a_n)))\\ &\quad \text{(by definition of } \equiv)\\ &= [\bar{f}](\pi(j_A(a_1)),\ldots,\pi(j_A(a_n)))\\ &\quad \text{(by definition of the quotient } \Sigma\text{-algebra).} \qquad \square \end{aligned} \tag{12}$$

Now I claim that $\mathbf{U}[V(A,F)]/\equiv$ is the free Σ-algebra in $\mathbf{U}$ generated by the Ω-algebra (A,F) in $\mathbf{V}$. To check this, let (B,G) be a Σ-algebra in $\mathbf{U}$, and let $\phi\colon (A,F) \to W(B,G)$ be a homomorphism of Ω-algebras. Then, $V(\phi)\colon A = V(A,F) \to V(W(B,G)) = U(B,G) = B$ is a set-theoretic map. Thus there is a unique homomorphism of Σ-algebras $\psi\colon \mathbf{U}[V(A,F)] \to (B,G)$ such that $U(\psi)\circ j_A = V(\phi)$. For each $f \in \Omega(n)$, $a_1,\ldots,a_n \in A$, one has

$$\begin{aligned}U(\psi)(j_A(f(a_1,\ldots,a_n))) &= V(\phi)(f(a_1,\ldots,a_n))\\ &= [\bar{f}](V(\phi)(a_1),\ldots,V(\phi)(a_n))\\ &\quad \text{(since } \phi \text{ is a homomorphism of } \Omega\text{-algebras)}\\ &= [\bar{f}](U(\psi)(j_A(a_1)),\ldots,U(\psi)(j_A(a_n)))\\ &= U(\psi)([\bar{f}](j_A(a_1),\ldots,j_A(a_n)))\\ &\quad \text{(because } \psi \text{ is a homomorphism of } \Sigma\text{-algebras).}\end{aligned} \tag{13}$$

Hence $\equiv$ is contained into $\ker\psi$. Thus, there exists a unique homomorphism of Σ-algebras $\tilde{\psi}\colon \mathbf{U}[V(A,F)]/\equiv \to B$ such that $\tilde{\psi}\circ\pi = \psi$. In particular, $V(W(\tilde{\psi})\circ i_{(A,F)}) = U(\tilde{\psi})\circ V(i_{(A,F)}) = U(\tilde{\psi})\circ U(\pi)\circ j_A = U(\tilde{\psi}\circ\pi)\circ j_A = U(\psi)\circ j_A = V(\phi)$. But as a forgetful functor from a variety to the category of sets, V is faithful, hence $W(\tilde{\psi})\circ i_A = \phi$ as expected. Uniqueness is obvious.

Whence the following is proved.

Theorem 18. *The notations and assumptions are as above. Let (A,F) be an Ω-algebra in the variety $\mathbf{V}$. Then, $\mathbf{U}[V(A,F)]/\equiv$ is the free Σ-algebra in the variety $\mathbf{U}$ generated by (A,F), and the insertion of generators is given by $i_{(A,F)}\colon (A,F) \to W(\mathbf{U}[V(A,F)]/\equiv)$.*

Remark 19. It is now easily seen that a left adjoint $L\colon \mathbf{V} \to \mathbf{U}$ of W is then given by $L(A,F) = \mathbf{U}[V(A,F)]/\equiv$ on objects, and given a homomorphism of Ω-algebras, in $\mathbf{V}$, $\phi\colon (A,F) \to (B,G)$, then $L(\phi)\colon \mathbf{U}[V(A,F)]/\equiv \to \mathbf{U}[V(B,G)]/\equiv$ is the unique homomorphism of Σ-algebras in $\mathbf{U}$ such that $L(\phi)\circ i_{(A,F)} = i_{(B,G)}\circ\phi$. The details are left to the reader.

In what follows the free object $\mathbf{U}[V(A,F)]/\equiv$ in $\mathbf{U}$, generated by (A,F) in $\mathbf{V}$, is denoted by $\mathbf{U}[A,F]$.

Remark 20. The reader could be confused by the use of the "polymorphic" notation $\mathbf{U}[-]$ for $\mathbf{U}[A, F]$ and for the free object $\mathbf{U}[A]$ in $\mathbf{U}$ generated by the set A. But the arguments for both occurrences of $\mathbf{U}[-]$ are not of the same type (an algebra for the former, and just a set for the latter). In what follows, to distinguish both notations, I follow a common practice from universal algebra: I use different calligraphies to denote an algebra and its carrier set. For instance, a typeface such as $\mathcal{A}$ will denote an algebra in some equational variety, while A will denote its carrier set, i.e., $\mathcal{A} = (A, F)$. Also $\mathfrak{g}$ will denote a Lie algebra, and g will be its carrier set.

Example 21.

1. Let **Grp** be the variety of all groups, and **Mon** be the variety of all monoids. Then, the *Grothendieck group completion* of a monoid $(M, *, e)$ (see, e.g., [27]) may be given as the free group $\mathbf{Grp}[M]$ factored out by the normal subgroup generated by $j_M(x)j_M(y)j_M(x * y)^{-1}$, $x, y \in M$, and by $j_M(e)$.
2. Let R be a commutative ring with a unit, and ${}_R\mathbf{Ass}_1$ be the variety of all unital (associative) R-algebras. Then, the monoid algebra of $(M, *, 1)$ may be described as the free associative algebra ${}_R\mathbf{Ass}_1[M]$ on the set M (which is more commonly denoted by $R\langle M\rangle$) divided out by the two-sided ideal generated by $j_M(x * y) - j_M(x)j_M(y)$, $j_M(1) - 1$, $x, y \in M$. This differs (but leads to isomorphic objects) from the usual way to define the monoid algebra as the free R-module on M with a multiplication inherited from that of the monoid by distributivity.
3. Now, let ${}_R\mathbf{Lie}$ be the variety of all Lie R-algebras. The algebraic functor from ${}_R\mathbf{Ass}_1$ to ${}_R\mathbf{Lie}$, that consists in considering any algebra as a Lie algebra under the commutator bracket, corresponds to the representation of ${}_R\mathbf{Lie}$ in ${}_R\mathbf{Ass}_1$ (see Example 16). Then, with Theorem 18, one gets a somewhat unusual construction for the universal enveloping algebra $\mathsf{U}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ (with carrier set g) as the quotient algebra of ${}_R\mathbf{Ass}_1[g]$ by the two-sided ideal generated by

$$
\begin{gathered}
j_g(x + y) - j_g(x) - j_g(y),\ j_g(\alpha x) - \alpha j_g(x),\ j_g(0),\\
j_g([x, y]) - j_g(x)j_g(y) + j_g(y)j_g(x)
\end{gathered}
$$

$x, y \in g, \alpha \in R$.

Remark 22. The notations and assumptions are as above. Let $(\mathbf{U}', U')$ (respectively, $(\mathbf{V}', V')$) be a concrete category concretely isomorphic to $\mathbf{U}$ (respectively, $\mathbf{V}$) by a concrete isomorphism $\Psi\colon (\mathbf{U}', U') \to (\mathbf{U}, U)$ (respectively, $\Phi\colon (\mathbf{V}', V') \to (\mathbf{V}, V)$). According to Lemma 14 the (concrete) functor $W'\colon \mathbf{U}' \to \mathbf{V}'$ given by $\Phi^{-1} \circ W \circ \Psi$, admits a left adjoint. One can equally well apply Theorem 18 to W'. Given a $\mathbf{V}'$-object v, the free $\mathbf{U}'$-object $\mathbf{U}'[v]$ generated by v is given by $\Psi^{-1}(\mathbf{U}[V(\Phi(v))]/\equiv)$, and the corresponding insertion of generators $j_v\colon v \to W'(\mathbf{U}'[v])$ is given by $\Phi^{-1}(i_{\Phi(v)})\colon v \to \Phi^{-1}(W(\mathbf{U}[V(\phi(v))]/\equiv)) = W'(\Psi^{-1}(\mathbf{U}[V(\phi(v))]/\equiv)) = W'(\mathbf{U}'[v])$. Once again in the rest of the paper I apply Theorem 18 in the case of concrete categories concretely isomorphic to varieties without further ado since we are on the safe side (the constructions are the same up to conjugation by isomorphisms).

5. Applications to "universal differential algebra"

In this section are applied the results from Section 4 in the realm of what I called "universal differential algebra". The goal is here to provide general functorial constructions in categories of differential algebras, some of them being already introduced in [25], in a uniform way. Of particular interest are the left adjoints of the functors occurring, as unnamed arrows, in the following (non-commutative) diagram (the denotations for the categories are introduced hereafter). For instance a left adjoint of the vertical arrow corresponds to the tensor algebra construction in differential algebra.

$$\begin{array}{ccc} {}_{(R,\partial)}\mathbf{DiffAss}_1 & \longrightarrow & {}_R\mathbf{Ass}_1 \\ \downarrow & & \\ {}_{(R,\partial)}\mathbf{DiffMod}_1 & \longrightarrow & {}_R\mathbf{Mod} \end{array} \tag{14}$$

5.1. Varieties of differential algebras

Let R be a commutative ring with a unit. (In what follows the base ring R will always be assumed associative, commutative and unital.) An *R-algebra* is a triple $((A, +, 0, \cdot), *)$ where $(A, +, 0, \cdot)$ is a R-module, and $*\colon A \times A \to A$ is a R-bilinear map. Hence associativity for the multiplication is not assumed. This defines a variety ${}_R\mathbf{Alg}$ (with the obvious homomorphisms of algebras). In more detail, one can consider the signature Σ_R, which depends on the ring R, with $\Sigma_R(0) = \{0\}$, $\Sigma_R(1) = \{-\} \cup \{\rho_\alpha\colon \alpha \in R\}$, $\Sigma_R(2) = \{+, *\}$, and the following (somewhat redundant) axioms (where α, β belong to R)

$$\begin{aligned}
&(x + y) + z = x + (y + z),\\
&x + y = y + x,\\
&x + 0 = x = 0 + x,\\
&x + (-x) = 0 = (-x) + x,\\
&\rho_\alpha(0) = 0,\\
&\rho_\alpha(x + y) = \rho_\alpha(x) + \rho_\alpha(y),\\
&\rho_\alpha(-x) = -\rho_\alpha(x),\\
&\rho_{\alpha+\beta}(x) = \rho_\alpha(x) + \rho_\beta(x),\\
&\rho_{-\alpha}(x) = -\rho_\alpha(x),\\
&\rho_0(x) = 0,
\end{aligned}$$

$$\begin{aligned}
&\rho_1(x) = x,\\
&x * (y + z) = (x * y) + (x * z),\\
&(x + y) * z = (x * y) + (x * z),\\
&\rho_\alpha(x * y) = (\rho_\alpha(x)) * y = x * (\rho_\alpha(y)). \qquad (15)
\end{aligned}$$

The variety of Σ_R-algebras satisfying the above axioms, seen as a category, is concretely isomorphic to the category of (not necessarily associative) usual R-algebras, and furthermore the latter is closed under homomorphic images, (cartesian) products and sub-algebras.

Similarly, one defines a *unital R-algebra* as an algebra over the signature Σ_R^1 which is obtained from Σ_R by adding a new function symbol 1 in $\Sigma_R(0)$ and which satisfies the additional equations $x * 1 = x = 1 * x$. This gives rise to the variety ${}_R\mathbf{Alg}_1$ (the homomorphisms are required to preserve the unit 1). By a *variety of R-algebras* (respectively, *unital R-algebras*) is meant a sub-variety of ${}_R\mathbf{Alg}$ (respectively, ${}_R\mathbf{Alg}_1$). In details: the objects of a variety of R-algebras are R-algebras that satisfy some given extra axioms. For instance, associative algebras ${}_R\mathbf{Ass}$, associative commutative algebras ${}_R\mathbf{Ass}_c$, Lie algebras ${}_R\mathbf{Lie}$ are varieties of R-algebras, while associative and unital algebras ${}_R\mathbf{Ass}_1$, and associative, commutative and unital algebras ${}_R\mathbf{Ass}_{c,1}$ are varieties of unital R-algebras.

From these varieties of algebras, one immediately obtains corresponding varieties of differential R-algebras, i.e., their counterparts in the differential algebra setting, as follows.

A part of the following material may be found in [16] (see also [18,19] in which differential rings usually come equipped with several commuting derivations instead of just one as in this contribution). A *differential ring* is a pair (R, ∂) where R is a unital ring and ∂ is a *derivation*, i.e., $\partial: R \to R$ is a group endomorphism of the underlying additive group of R, and it satisfies *Leibniz rule* $\partial(\alpha\beta) = \partial(\alpha)\beta + \alpha\partial(\beta)$, $a, b \in R$. In particular, $\partial(1) = 0$.

Remark 23. Every ring R admits the zero map $a \in R \mapsto 0$ as a derivation, known as the *trivial* (or *zero*) *derivation*. This is the unique derivation on $\mathbb{Z}$.

Let us consider a commutative differential ring (R, ∂), i.e., a differential ring which is commutative as a ring. Let ${}_R\mathbf{V}$ (respectively, ${}_R\mathbf{V}_1$) be a variety of R-algebras (respectively, unital R-algebras). One defines the variety ${}_{(R,\partial)}\mathbf{DiffV}$ (respectively, ${}_{(R,\partial)}\mathbf{DiffV}_1$) of *$(R, \partial)$-differential (unital) algebras* in ${}_R\mathbf{V}$ (respectively, ${}_R\mathbf{V}_1$). Its objects are pairs $(\mathcal{A}, d)$ where $\mathcal{A} = ((A, +, 0, \cdot), *)$ (respectively, $\mathcal{A} = ((A, +, 0, \cdot), *, 1)$) is a R-algebra in ${}_R\mathbf{V}$ (respectively, in ${}_R\mathbf{V}_1$), and $d: (A, +, 0) \to (A, +, 0)$ is *(R, ∂)-derivation*, i.e., an additive map that satisfies the Leibniz rule

$$d(x * y) = d(x) * y + x * d(y)$$

and also the equation $d(\alpha \cdot x) = \partial(\alpha) \cdot x + \alpha \cdot d(x)$, $\alpha \in R$, $x, y \in A$. (One observes that if ${}_R\mathbf{V}_1$ is a variety of R-algebras with a unit, then $d(1) = 0$ because $d(1) = d(1 * 1) = d(1)*1+1*d(1) = d(1)+d(1)$.) If ∂ is the zero derivation on R, then any $(R, 0)$-derivation is a R-linear map which satisfies Leibniz rule. A *homomorphism of differential algebras* is just a homomorphism of algebras that commutes with the derivations (and that preserves the units if any.)

Remark 24. The category ${}_R\mathbf{Mod}$ of R-modules may be identified with a variety of R-algebras, namely the variety ${}_R\mathbf{Triv}$ of R-algebras with the zero multiplication (for all x, y, $x*y = 0$). Hence one can consider ${}_{(R,\partial)}\mathbf{DiffMod}$, identified with ${}_{(R,\partial)}\mathbf{DiffTriv}$, as the variety of differential R-modules. Its objects are just usual R-modules M together with a group endomorphism d (for their underlying additive group structure) such that $d(\alpha \cdot x) = \partial(\alpha) \cdot d(x) + \alpha \cdot d(x)$, $\alpha \in R$, $x \in M$. In particular, if ∂ is the zero derivation on R, then the objects of ${}_{(R,0)}\mathbf{DiffMod}$ are just modules together with a module endomorphism.

For an algebra $\mathcal{A} = ((A, +, 0, \cdot), *)$ (respectively, unital algebra $\mathcal{A} = ((A, +, 0, \cdot), *, 1)$) in a variety ${}_R\mathbf{V}$ (respectively, ${}_R\mathbf{V}_1$) of R-algebras (respectively, unital R-algebras) a *(two-sided) ideal* is a R-sub-module I such that for each $x \in A$, $y \in I$, $x * y, y * x \in I$. The notion of the *ideal generated by a sub-set* $E \subseteq A$ is obvious. Given an ideal I, A/I inherits from $\mathcal{A}$ a structure of a R-algebra which is denoted by $\mathcal{A}/I$, and the canonical epimorphism $\pi: A \to A/I$ becomes a homomorphism of R-algebras. Moreover, the algebra $\mathcal{A}/I$ belongs to the variety ${}_R\mathbf{V}$ (respectively, ${}_R\mathbf{V}_1$). This is clear because there is a one–one correspondence between ideals and congruences (see Definition 6), and any variety is closed under quotients (by a congruence).

Indeed, to any congruence on an algebra is associated its equivalence class of zero, that turns to be a two-sided ideal (because if x is congruent to 0, then, by definition of a congruence and because by bilinearity of the multiplication 0 is a two-sided absorbing element, for every y, $x * y$ is congruent to $0 = 0 * y$, $y * x$ is congruent to $0 = y * 0$), and, conversely, any two-sided ideal I defines a congruence by setting that x and y are equivalent whenever $x - y \in I$ (this of course defines an equivalence relation, which furthermore is a congruence since $x - y \in I$ and $x' - y' \in I$ leads to $x * x' - y * y' \in I$). Of course, the (usual) kernel of a homomorphism of algebras provides an ideal, and all ideals are obtained in this way. It may also be noticed that the notion of kernel from universal algebra (see Remark 8) coincides with the usual one since, for a linear map ϕ, $\phi(x) = \phi(y)$ if, and only if, $\phi(x - y) = 0$.

In the same way for a differential algebra $(\mathcal{A}, d)$ in ${}_{(R,\partial)}\mathbf{DiffV}$ (or in ${}_{(R,\partial)}\mathbf{DiffV}_1$) one can also talk about a *(two-sided) differential ideal* as a usual ideal I of the R-algebra $\mathcal{A}$ such that $d(I) \subseteq I$. Given a set $E \subseteq A$, the *differential ideal* $\langle E \rangle$ *generated by* E is just the (algebraic) two-sided ideal generated by $\{d^n(x): x \in I,\ n \geq 0\}$. Again A/I becomes a (R, ∂)-differential algebra in ${}_{(R,\partial)}\mathbf{DiffV}$ (or in ${}_{(R,\partial)}\mathbf{DiffV}_1$), with the (R, ∂)-derivation $\bar{d}(x + I) = d(x) + I$, and the canonical epimorphism commutes with the derivations, whence is a homomorphism of differential algebras in ${}_{(R,\partial)}\mathbf{DiffV}$ (or in ${}_{(R,\partial)}\mathbf{DiffV}_1$).

There is an obvious forgetful functor $dV: {}_{(R,\partial)}\mathbf{DiffV} \to {}_R\mathbf{V}$ (and, respectively, from ${}_{(R,\partial)}\mathbf{DiffV}_1$ to ${}_R\mathbf{V}_1$).

Remark 25. dV is a faithful functor.

The functor dV makes commute a diagram such as Diagram (10). Hence the construction from Section 4 may be applied here, so that dV admits a left adjoint, or alternatively, for any algebra in ${}_R\mathbf{V}$ there exists a free object in ${}_{(R,\partial)}\mathbf{DiffV}$ generated by the algebra (a corresponding result of course holds for ${}_R\mathbf{V}_1$ instead of ${}_R\mathbf{V}$). Let $\mathcal{A} = ((A, +, 0, \cdot), *)$ be a R-algebra in ${}_R\mathbf{V}$. Then, by Theorem 18, ${}_{(R,\partial)}\mathbf{DiffV}[\mathcal{A}] = {}_{(R,\partial)}\mathbf{DiffV}[A]/I$ where

$$\begin{aligned} I = \langle & j_A(x+y) - j_A(x) - j_A(y),\ j_A(0),\ j_A(\alpha \cdot x) - \alpha \cdot j_A(x), \\ & j_A(x * y) - J_A(x) * j_A(y) : x, y \in A,\ \alpha \in R \rangle \end{aligned}$$

(in case of a variety ${}_R\mathbf{V}_1$ of unital algebras, one adds $j_A(1)-1$ as a generator for the ideal), where $j_A: A \to {}_{(R,\partial)}\mathbf{DiffV}[A]$ is the insertion of generators, and ${}_{(R,\partial)}\mathbf{DiffV}[X]$ is the free differential algebra in the variety $\mathbf{V}$ generated by a set X, which exists since ${}_{(R,\partial)}\mathbf{DiffV}$ is an equational variety of algebras (see Section 3).

Example 26.

1. Let us consider the variety ${}_R\mathbf{Lie}$ of all Lie R-algebras. Hence ${}_{(R,\partial)}\mathbf{DiffLie}$ is the variety of all (R, ∂)-differential Lie algebras, i.e., pairs $((\mathfrak{g}, [-, -]), d)$ where $(\mathfrak{g}, [-, -])$ is a Lie R-algebra, and $d: \mathfrak{g} \to \mathfrak{g}$ is an abelian group morphism such that $d([x, y]) = [d(x), y] + [x, d(y)]$ and $d(\alpha \cdot x) = \partial(\alpha) \cdot x + \alpha \cdot d(x)$. It follows that the free (R, ∂)-differential Lie algebra generated by the Lie algebra $(\mathfrak{g}, [-, -])$, with underlying set g, is given by ${}_{(R,\partial)}\mathbf{DiffLie}[\mathfrak{g}, [-, -]] = {}_{(R,\partial)}\mathbf{DiffLie}[g]/I$ where

$$\begin{aligned} I = \langle & j_g(x+y) - j_g(x) - j_g(y),\ j_g(0),\ j_g(\alpha \cdot x) - \alpha \cdot j_g(x), \\ & j_g([x, y]) - [J_g(x), j_g(y)] : x, y \in g,\ \alpha \in R \rangle \end{aligned}$$

   (recall here that g denotes the underlying set of the module $\mathfrak{g}$; see Remark 20), according to Theorem 18.
2. Similarly, let us consider ${}_R\mathbf{Ass}_1$ (respectively, ${}_R\mathbf{Ass}_{c,1}$) the variety of all unital and associative (and, respectively, commutative) R-algebras. Given a unital and associative (and, respectively, commutative) algebra $\mathcal{A}$, with underlying set A, one gets the free (R, ∂)-differential unital and associative (and, respectively, commutative) algebra as the quotient algebra ${}_{(R,\partial)}\mathbf{DiffAss}_1[A]/I$ (respectively, ${}_{(R,\partial)}\mathbf{DiffAss}_{c,1}[A]/I$), where I is the two-sided differential ideal generated by

$$j_A(x+y) - j_A(x) - j_A(y), j_A(\alpha x) - \alpha j_A(x), j_A(0), j_A(1) - 1, j_A(xy) - j_A(x) j_A(y)$$

   for $x, y \in A$, $\alpha \in R$. This is called the *differential envelope* of $\mathcal{A}$ in [25].

3. A differential ideal in ${}_{(R,\partial)}\mathbf{DiffMod}$ (see Remark 24) turns to be a sub-module closed under the derivation, i.e., a *differential sub-module*. Hence one can also consider the free (R,∂)-differential module ${}_{(R,\partial)}\mathbf{DiffMod}[\mathcal{M}]$ generated by a R-module $\mathcal{M} = (M,+,0,\cdot)$ as ${}_{(R,\partial)}\mathbf{DiffMod}[\mathcal{M}] = {}_{(R,\partial)}\mathbf{DiffMod}[M]/N$ where N is the differential sub-module generated by

$$j_M(x+y) - j_M(x) - j_M(y),\ j_M(0)\, j_M(\alpha\cdot x) - \alpha j_M(x)$$

$x, y \in M$, $\alpha \in R$. In Subsection 5.2 is provided a different construction for this object.

In the particular case of a variety of differential algebras in which the derivation on the ground ring R is zero, one may also consider differential algebras in which $d = 0$. More precisely if $(R,\partial) = (R,0)$, then there is an obvious $(R,0)$-derivation for each algebra in ${}_R\mathbf{V}$, namely the zero derivation. This means that there exists also an obvious *inclusion functor* $J\colon {}_R\mathbf{V} \to {}_{(R,0)}\mathbf{DiffV}$ given at the level of objects by $J(\mathcal{A}) = (\mathcal{A}, 0)$ since 0 is a $(R,0)$-derivation, and for a homomorphism ϕ, one has $J(\phi) = \phi$ since any homomorphism of algebras maps 0 to 0. This is a full embedding (i.e., a full and faithful functor, and injective on objects). One observes that J is an algebraic functor (since it commutes with the forgetful functors), hence it admits a left adjoint, i.e., ${}_R\mathbf{V}$ is a reflective sub-category of ${}_{(R,0)}\mathbf{DiffV}$ (see [21]).

The construction of a left adjoint of J goes as follows: let $(\mathcal{A}, d)$ be an object of ${}_{(R,0)}\mathbf{DiffV}$ (hence $\mathcal{A} = ((A,+,0,\cdot),*)$ is an object of ${}_R\mathbf{V}$). Let I_d be the (algebraic) two-sided ideal generated by $im(d)$ (where d is seen as a R-linear map) in $\mathcal{A}$. Hence, $\mathcal{A}/I_d$ is a member of ${}_R\mathbf{V}$ and the natural projection $\pi\colon \mathcal{A} \to \mathcal{A}/I_d$ is a homomorphism of algebras. Let $\mathcal{B}$ be an algebra in ${}_R\mathbf{V}$, and let $\phi\colon (\mathcal{A}, d) \to (\mathcal{B}, 0)$ be a homomorphism of $(R,0)$-differential algebras in ${}_R\mathbf{V}$. Because $\phi \circ d = 0$, it follows that ϕ factors uniquely through π as a homomorphism of algebras $\hat{\phi}\colon \mathcal{A}/I_d \to \mathcal{B}$ such that $\hat{\phi} \circ \pi = \phi$. Finally one gets:

Proposition 27. *For every $(R,0)$-differential algebra $(\mathcal{A}, d)$ in ${}_{(R,0)}\mathbf{DiffV}$, there exists a free R-algebra in ${}_R\mathbf{V}$ generated by $(\mathcal{A}, d)$, namely $\mathcal{A}/I_d$, where I_d is as above.*

Remark 28. A proposition corresponding to Proposition 27 of course holds for ${}_R\mathbf{V}_1$ instead of ${}_R\mathbf{V}$.

The inclusion functor J also admits a right adjoint, which merely is the usual "ring of constants" construction. Indeed, let us consider an object $(\mathcal{A}, d)$ in ${}_{(R,0)}\mathbf{DiffV}$, and let us define $\mathbf{Const}(\mathcal{A}, d) = \ker(d)$. Then, $\mathbf{Const}(\mathcal{A}, d)$ is a sub-R-algebra of $\mathcal{A}$, whence also a member of ${}_{(R,0)}\mathbf{DiffV}$, and the canonical inclusion $incl\colon \mathbf{Const}(\mathcal{A}, d) \hookrightarrow \mathcal{A}$ is a homomorphism of algebras. Now, let $\mathcal{B}$ be an algebra in ${}_R\mathbf{V}$, and let $\phi\colon (\mathcal{B}, 0) \to (\mathcal{A}, d)$ be a homomorphism of $(R,0)$-differential algebras. Since $d \circ \phi = 0$, $im(\phi) \subseteq \mathbf{Const}(\mathcal{A}, d)$, therefore there exists a unique algebra map $\bar{\phi}\colon \mathcal{B} \to \mathbf{Const}(\mathcal{A}, d)$ such that $incl \circ \bar{\phi} = \phi$.

Proposition 29. *For every $(R,0)$-differential algebra $(\mathcal{A},d)$ in ${}_{(R,0)}\mathbf{DiffV}$, there exists a cofree R-algebra in ${}_R\mathbf{V}$ cogenerated by $(\mathcal{A},d)$, namely $\mathbf{Const}(\mathcal{A},d)$ as above.*

Remark 30. A proposition corresponding to Proposition 29 of course holds for ${}_R\mathbf{V}_1$ instead of ${}_R\mathbf{V}$.

From a category-theoretic perspective, Propositions 27 and 29 mean that ${}_R\mathbf{V}$ is both a reflective and coreflective sub-category of ${}_{(R,0)}\mathbf{DiffV}$.

One finally mentions a last construction – from [14] – also in the case where $(R,\partial) = (R,0)$ that plays no role in the sequel but which is given for the sake of completeness. Let $\mathcal{A}$ be an associative R-algebra with a unit (with underlying set A). The abelian group $A^{\mathbb{N}}$ (under point-wise addition) admits a ring structure given by

$$(fg)(n) = \sum_{k=0}^{n} \binom{n}{k} f(n-k)g(k)$$

$f,g \in A^{\mathbb{N}}$, $n \in \mathbb{N}$. Using the ring map $R \to A^{\mathbb{N}}$ sending 1 to δ_0 ($\delta_0(n) = 0$ if $n \neq 0$, and $\delta_0(0) = 1$), one obtains a structure of an associative R-algebra with a unit called the *R-algebra of Hurwitz series over $\mathcal{A}$* (see [17]) denoted by $\mathcal{A}^{\mathbb{N}}$. It admits a $(R,0)$-derivation given by $(d(f))(n) = f(n+1)$, $f \in \mathcal{A}^{\mathbb{N}}$.

Proposition 31. *(See [14, Proposition 2.8].) $(\mathcal{A}^{\mathbb{N}},d)$ is the cofree $(R,0)$-differential algebra on $\mathcal{A}$, i.e., for every $(R,0)$-differential algebra $(\mathcal{B},e)$ and every homomorphism of R-algebras $\phi\colon \mathcal{B} \to \mathcal{A}$, there is a unique homomorphism of $(R,0)$-differential algebras $\hat{\phi}\colon (\mathcal{B},e) \to (\mathcal{A}^{\mathbb{N}},d)$ such that $\epsilon_{\mathcal{A}} \circ DiffAss_1(\hat{\phi}) = \phi$, where $\epsilon_{\mathcal{A}}(f) = f(0)$. This equivalently means that the obvious forgetful functor $DiffAss_1\colon {}_{(R,0)}\mathbf{DiffAss}_1 \to {}_R\mathbf{Ass}_1$ admits a right adjoint.*

Proof. Let us define $\hat{\phi}(b)(n) = \phi(e^n(b))$, $b \in B$, $n \in \mathbb{N}$. One has $\hat{\phi}(e(b))(n) = \phi(e^n(e(b))) = \phi(e^{n+1}(b)) = \hat{\phi}(b)(n+1) = d(\hat{\phi}(b))(n)$. The fact that $\hat{\phi}$ is a R-algebra map is easily checked, as is proved uniqueness. $\square$

5.2. The free differential module generated by a module

One can form the free (R,∂)-differential module on a set X as follows: as a R-module it is just $R \otimes_{\mathbb{Z}} (\mathbb{Z}X \times \mathbb{N})$ (where RX denotes the free R-module on a set X for any commutative ring R, thus $\mathbb{Z}X \times \mathbb{N}$ is the free abelian group on the set $X \times \mathbb{N}$) with R-action given by $\alpha \cdot (\beta \otimes (x,i)) = (\alpha\beta) \otimes (x,i)$, and with the derivation given by $d(\alpha \otimes (x,i)) = \partial(\alpha) \otimes (x,i) + \alpha \otimes (x,i+1)$. One has

$$\begin{aligned} d(\alpha \cdot (\beta \otimes (x,i))) &= d((\alpha\beta) \otimes (x,i)) \\ &= \partial((\alpha\beta) \otimes (x,i)) + (\alpha\beta) \otimes (x,i+1) \\ &= (\partial(\alpha)\beta) \otimes (x,i) + (\alpha\partial(\beta)) \otimes (x,i) + (\alpha\beta) \otimes (x,i+1) \end{aligned}$$

$$= \partial(\alpha) \cdot (\beta \otimes (x,i)) + \alpha \cdot (\partial(\beta) \otimes (x,i) + \beta \otimes (x,i+1))$$
$$= \partial(\alpha) \cdot (\beta \otimes (x,i)) + \alpha \cdot d(\beta \otimes (x,i)). \tag{16}$$

Let us check freeness of this construction.

Lemma 32. *The R-module $R \otimes_{\mathbb{Z}} (\mathbb{Z}X \times \mathbb{N})$, with the (R, ∂)-derivation d as above, is the free (R, ∂)-differential module generated by the set X.*

Proof. Let $(\mathcal{M}, e)$ be a (R, ∂)-differential module, with $\mathcal{M} = (M, +, 0, \cdot)$, and let $\phi : X \to M$ be a set-theoretic map. By recurrence one defines $\phi_1 : X \times \mathbb{N} \to M$ by $\phi_1(x, 0) = \phi(x)$ and $\phi_1(x, i+1) = e(\phi_1(x,i))$, $i \geq 0$. Hence $\phi_1(x,i) = e^i(\phi(x))$. Let $\phi_2 : \mathbb{Z}X \times \mathbb{N} \to (M, +, 0)$ be the unique $\mathbb{Z}$-module map that extends ϕ_1. Let $\phi_3 : R \times (\mathbb{Z}X \times \mathbb{N}) \to (M, +, 0)$ be the bi-additive (i.e., $\mathbb{Z}$-bilinear) map given by $\phi_3(\alpha, (x,i)) = \alpha\phi_2(x,i)$. Hence it induces a unique abelian group homomorphism $\phi_4 : R \otimes_{\mathbb{Z}} (\mathbb{Z}X \times \mathbb{N}) \to (M, +, 0)$ by $\phi_4(\alpha \otimes (x,i)) = \phi_3(\alpha, (x,i))$. It is actually R-linear. Indeed,

$$\begin{aligned}\phi_4(\alpha \cdot (\beta \otimes (x,i))) &= \phi_4((\alpha\beta) \otimes (x,i)) \\ &= (\alpha\beta)\phi_2(x,i) \\ &= \alpha(\beta\phi_2(x,i)) \\ &= \alpha\phi_4(\beta \otimes (x,i)).\end{aligned} \tag{17}$$

This map commutes with the derivations. Indeed,

$$\begin{aligned}\phi_4(d(\alpha \otimes (x,i))) &= \phi_4(\partial(\alpha) \otimes (x,i)) + \phi_4(\alpha \otimes (x,i+1)) \\ &= \partial(\alpha)d^i(\phi(x)) + \alpha e^{i+1}(\phi(x)) \\ &= e(\alpha e^i(\phi(x))) \\ &= e(\phi_4(\alpha \otimes (x,i))).\end{aligned} \tag{18}$$

This map satisfies $\phi_4 \circ j_X = \phi$, where $j_X : X \to R \otimes_{\mathbb{Z}} (\mathbb{Z}X \times \mathbb{N})$ is given by $j_X(x) = 1 \otimes (x, 0)$. Let $\psi : (R \otimes_{\mathbb{Z}} (\mathbb{Z}X \times \mathbb{N}), d) \to (\mathcal{M}, e)$ be a homomorphism of (R, ∂)-differential modules such that $\psi(j_X(x)) = \phi(x)$. Then,

$$\begin{aligned}\psi(\alpha \otimes (x,i)) &= \psi(\alpha(1 \otimes (x,i))) \\ &= \alpha\psi(1 \otimes (x,i)) \\ &= \alpha\psi(d^i(1 \otimes (x,0))) \\ &= \alpha e^i(\psi(1 \otimes (x,0))) \\ &= \alpha e^i \phi(x) \\ &= \phi_4(\alpha \otimes (x,i)).\end{aligned} \tag{19}$$

Hence ϕ_4 is uniquely determined by ϕ and by its defining property. □

Remark 33. Of course for any set X, as R-modules, $R \otimes_{\mathbb{Z}} \mathbb{Z}X \simeq RX$ by $x \mapsto 1 \otimes x$, $x \in X$. Under such an isomorphism, the free (R, ∂)-differential module generated by a set X is $RX \times \mathbb{N}$ together with the derivation $d(\alpha(x,i)) = \partial(\alpha)(x,i) + \alpha(x,i+1)$.

Remark 34. Likewise one can construct the free (R, ∂)-differential commutative and unital algebra ${}_{(R,\partial)}\mathbf{DiffAss}_{c,1}[X]$ generated by a set X as $R \otimes_{\mathbb{Z}} \mathbb{Z}[X \times \mathbb{N}] \simeq R[X \times \mathbb{N}]$ (where $R[X]$ is the polynomial algebra in the variables in X) with the derivation $d(\alpha(x,i)) = \partial(\alpha)(x,i) + \alpha(x,i+1)$. Of course, one recovers the usual differential polynomial algebra $R\{X\}$ with (commutative) variables in X. Moreover, the free (R, ∂)-differential ideal generated by the set X may be identified with those linear polynomials in $R\{X\}$ without constant terms.

Using Lemma 32 (and its proof) and Theorem 18 one can form the free (R, ∂)-differential module generated by a R-module $\mathcal{M} = (M, +, 0, \cdot)$ as follows: ${}_{(R,\partial)}\mathbf{DiffMod}[\mathcal{M}] := ((R \otimes_{\mathbb{Z}} (\mathbb{Z}M \times \mathbb{N}), d)/\langle 1 \otimes (x+y, 0) - 1 \otimes (x, 0) - 1 \otimes (y, 0),\ 1 \otimes (0,0),\ 1 \otimes (\alpha x, 0) - \alpha \otimes (x, 0) \colon x, y \in M,\ \alpha \in R\rangle$ where $\langle X \rangle$ denotes the (R, ∂)-differential sub-module generated by X, i.e., the usual R-sub-module generated by $d^n(t)$ for all $t \in X$, $n \geq 0$.

5.3. Tensor and symmetric (differential) algebra functors

The obvious forgetful functor $W \colon {}_{(R,\partial)}\mathbf{DiffAss}_1 \to {}_{(R,\partial)}\mathbf{DiffMod}$ satisfies the assumptions from Section 4, hence it admits a left adjoint that may be described by its action on objects as ${}_{(R,\partial)}\mathbf{DiffAss}_1[M]/I$ where

$$\begin{aligned} I = \langle & j_M(x+y) - j_M(x) - j_M(y), j_M(\alpha x) - \alpha j_M(x), j_M(0), \\ & j_M(d(x)) - (j_M(x))' \colon x, y \in M, \alpha \in R\rangle \end{aligned}$$

for $(\mathcal{M}, d) = ((M, +, 0, \cdot), d)$ an (R, ∂)-differential module (and by t' is denoted the derivative of t in ${}_{(R,\partial)}\mathbf{DiffAss}_1[M]$). This left adjoint corresponds, in the category of differential algebras, to the usual tensor algebra functor in the category of non-differential algebras. Some related constructions are provided in [9,13,15].

There is another way, more accurate, to define it, that makes possible to directly relate it with the usual tensor algebra construction. But in order to describe it one first needs some notations. First of all, one denotes by $|\mathcal{M}|$ the underlying abelian group of a R-module $\mathcal{M}$. Secondly, the generators of $A \otimes_{\mathbb{Z}} B$ (respectively, $\mathcal{M} \otimes_R \mathcal{N}$) for abelian groups A, B (respectively, R-modules $\mathcal{M}, \mathcal{N}$) are denoted by $x \otimes_{\mathbb{Z}} y$ (respectively, $x \otimes_R y$) to avoid any confusion. These notations are extended in an obvious way for n-fold tensor products (not only 2-fold ones).

Finally, one recalls that for $\mathcal{M}_i$, $i = 1, \ldots, n$, R-modules, the underlying abelian group $|\mathcal{M}_1 \otimes_R \cdots \otimes_R \mathcal{M}_n|$ of the R-module $\mathcal{M}_1 \otimes_R \cdots \otimes_R \mathcal{M}_n$ is given by the quotient of the abelian group $|\mathcal{M}_1| \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} |\mathcal{M}_n|$ by the subgroup generated by

$$x_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} (\alpha x_i) \otimes_{\mathbb{Z}} \cdots_{\mathbb{Z}} \otimes_{\mathbb{Z}} x_n - x_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} (\alpha x_j) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} x_n$$

for all $x_k \in |\mathcal{M}_k|$, $k = 1, \ldots, n$, for all $1 \leq i \neq j \leq n$, and for all $\alpha \in R$ (see e.g. [7, Chap. II]). One denotes by *can* the associated canonical epimorphism.

Let $(\mathcal{M}, d)$ be a (R, ∂)-differential module. Let $v_1, \ldots, v_n \in M$, and let us consider the map $D(v_1, \ldots, v_n) = d(v_1) \otimes_{\mathbb{Z}} v_2 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} v_n + v_1 \otimes_{\mathbb{Z}} d(v_2) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} v_n + \cdots + v_1 \otimes_{\mathbb{Z}} v_2 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} d(v_n)$. It is a $\mathbb{Z}$-multilinear map hence gives rise to an endomorphism, again denoted by D, of the abelian group $\underbrace{|\mathcal{M}| \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} |\mathcal{M}|}_{n \text{ factors}}$. It is of course a $(\mathbb{Z}, 0)$-derivation. One observes easily that $D(x_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} (\alpha x_i) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} x_n) = D(x_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} (\alpha x_j) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} x_n)$, hence there is a unique endomorphism $\tilde{D}$ of the underlying abelian group of $\underbrace{\mathcal{M} \otimes_R \cdots \otimes_R \mathcal{M}}_{n \text{ factors}}$ such that $\tilde{D} \circ can = can \circ D$. One also extends $\tilde{D}$ on R by setting $\tilde{D}(\alpha) = \partial(\alpha)$. It follows that one obtains a homomorphism of the underlying abelian group of the tensor algebra $\mathsf{T}(\mathcal{M})$ on $\mathcal{M}$ (this is due to the fact: $|\mathcal{M}_1| \oplus \cdots \oplus |\mathcal{M}_n| \simeq |\mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n|$ for all R-modules $\mathcal{M}_i$, $i = 1, \ldots, n$), again denoted by $\tilde{D}$. It remains to make sure that $\tilde{D}$ is a (R, ∂)-derivation. Actually it is easy to see that $\tilde{D}$ is a $(\mathbb{Z}, 0)$-derivation of the carrier abelian group $|\mathsf{T}(\mathcal{M})|$ of $\mathsf{T}(\mathcal{M})$ and the fact that $\tilde{D}(\alpha w) = \partial(\alpha) w + \alpha \tilde{D}(w)$ follows from the definition of $\tilde{D}$.

Remark 35. Let $j: \mathcal{M} \to \mathsf{T}(\mathcal{M})$ be the canonical injection. Then, j commutes with the derivations (since $\tilde{D}(j(x)) = D(j(x)) = j(d(x))$), hence defines a homomorphism of differential modules from $(\mathcal{M}, d)$ to $(\mathsf{T}(\mathcal{M}), \tilde{D})$.

My claim is that $(\mathsf{T}(\mathcal{M}), \tilde{D})$ is the free (R, ∂)-differential associative and unital algebra generated by $(\mathcal{M}, d)$, called the *tensor* (R, ∂)*-differential algebra* on $(\mathcal{M}, d)$.

Proposition 36. *The free (R, ∂)-differential (associative and unital) algebra ${}_{(R,\partial)}\mathbf{DiffAss}_1[\mathcal{M}, d]$ generated by a (R, ∂)-differential module $(\mathcal{M}, d)$ is* $(\mathsf{T}(\mathcal{M}), \tilde{D})$.

Proof. Let $\phi: (\mathcal{M}, d) \to W(\mathcal{A}, e)$ be a homomorphism of differential modules (hence just a R-linear map that commutes with the derivations). Let $\hat{\phi}: \mathsf{T}(\mathcal{M}) \to \mathcal{A}$ be the unique algebra map that extends ϕ. One has

$$\begin{aligned} \hat{\phi}(\tilde{D}(x_1 \otimes_R \cdots \otimes_R x_n)) &= \hat{\phi}(d(x_1) \otimes_R \cdots \otimes_R x_n) + \cdots + \hat{\phi}(x_1 \otimes_R \cdots \otimes_R d(x_n)) \\ &= \phi(d(x_1)) \cdots \phi(x_n) + \cdots + \phi(x_1) \cdots \phi(d(x_n)) \\ &= e(\phi(x_1)) \cdots \phi(x_n) + \cdots + \phi(x_1) \cdots e(\phi(x_n)) \\ &= e(\hat{\phi}(x_1 \cdots x_n)). \qquad \square \end{aligned} \tag{20}$$

One remarks that $\tilde{D}$ commutes with the symmetry operators U_σ in $\mathsf{T}(\mathcal{M})$ (such an operator is defined by $U_\sigma: x_1 \otimes_R \cdots \otimes_R x_n \to x_{\sigma(1)} \otimes_R \cdots \otimes_R x_{\sigma(n)}$ for some permutation $\sigma \in \mathfrak{S}_n$ and thus consists in a permutation of the variables), i.e., $U_\sigma(\tilde{D}(x_1 \otimes_R \cdots \otimes_R$

$x_n)) = \tilde{D}(x_{\sigma(1)} \otimes_R \cdots \otimes_R x_{\sigma(n)})$ for each $\sigma \in \mathfrak{S}_n$. Hence it factors through the symmetric algebra $\mathsf{S}(\mathcal{M})$ on $\mathcal{M}$ to provide an abelian group endomorphism still denoted $\tilde{D}$. The resulting map is easily shown to be a (R,∂)-derivation.

Remark 37. For $(\mathcal{M}, d)$ a (R,∂)-differential module, the canonical projection $\pi\colon \mathsf{T}(\mathcal{M}) \to \mathsf{S}(\mathcal{M})$ of course commutes with the derivations and thus defines a homomorphism of (R,∂)-differential algebras. It then follows that the canonical injection $\mathcal{M} \to \mathsf{S}(\mathcal{M})$ also commutes with the derivations hence defines a homomorphism of differential modules, since it is given as the composition of π and the canonical injection j from $\mathcal{M}$ into $\mathsf{T}(\mathcal{M})$.

It is left to the reader to show that $(\mathsf{S}(\mathcal{M}), \tilde{D})$ is the free (R,∂)-differential (associative, unital) commutative algebra on $\mathcal{M}$, called the *symmetric (R,∂)-differential algebra* on $(\mathcal{M}, d)$.

Remark 38. The above extension of a (R,∂)-derivation to the tensor and the symmetric algebras is a generalization of the extension of an endomorphism to a $(R,0)$-derivation from [6, Lemme 4, p. 35].

It is clear by composition of adjoints that if X is any set, then the symmetric differential algebra $(\mathsf{S}({}_{(R,\partial)}\mathbf{DiffMod}[X]), \tilde{D})$ on the free differential module ${}_{(R,\partial)}\mathbf{DiffMod}[X]$ on X is nothing else than the algebra of differential polynomials $R\{X\}$ with commuting variables in X (see Remark 34).

Of course, similarly to the non-differential case, $(\mathcal{M}, d)$ both embeds into $(\mathsf{T}(\mathcal{M}), \tilde{D})$ and into $(\mathsf{S}(\mathcal{M}), \tilde{D})$ as a differential sub-module using the insertions of generators j and $\pi \circ j$. One observes that the following diagrams commute (where, I recall that $dV\colon {}_{(R,\partial)}\mathbf{DiffV} \to {}_R\mathbf{V}$ denotes the obvious forgetful functor for each variety of R-algebras ${}_R\mathbf{V}$, and similarly for a variety ${}_R\mathbf{V}_1$ of unital R-algebras; see Subsection 5.1).

$$\begin{array}{ccc} {}_{(R,\partial)}\mathbf{DiffMod} & \xrightarrow{\mathsf{T}} & {}_{(R,\partial)}\mathbf{DiffAss}_1 \\ \downarrow{\scriptstyle dMod} & & \downarrow{\scriptstyle dAss} \\ {}_R\mathbf{Mod} & \xrightarrow[\mathsf{T}]{} & {}_R\mathbf{Ass}_1 \end{array} \tag{21}$$

and

$$\begin{array}{ccc} {}_{(R,\partial)}\mathbf{DiffMod} & \xrightarrow{\mathsf{S}} & {}_{(R,\partial)}\mathbf{DiffAss}_{c,1} \\ \downarrow{\scriptstyle dMod} & & \downarrow{\scriptstyle dAss_c} \\ {}_R\mathbf{Mod} & \xrightarrow[\mathsf{S}]{} & {}_R\mathbf{Ass}_{c,1} \end{array} \tag{22}$$

Hence it makes sense to say that the tensor (symmetric) differential algebra functor is obtained by lifting the usual tensor (symmetric) algebra functor to the categories of differential algebras.

5.4. Monoid and group differential algebra

Let us consider the following commutative diagram of obvious forgetful functors (M forgets the additive structure, and $dAss$ forgets the derivation).

$$\begin{array}{ccccc} {}_{(R,\partial)}\mathbf{DiffAss}_1 & \xrightarrow{dAss} & {}_R\mathbf{Ass}_1 & \xrightarrow{M} & \mathbf{Mon} \\ & \searrow^{DiffAss_1} & \downarrow^{Ass_1} & \swarrow_{Mon} & \\ & & \mathbf{Set} & & \end{array} \tag{23}$$

The differential envelope ${}_{(R,\partial)}\mathbf{DiffAss}_1[R[\mathcal{M}]]$ (see Example 26.2) of the usual monoid algebra $R[\mathcal{M}]$ of a monoid $\mathcal{M}$ (a left adjoint of M) is, by composition of left adjoints, a left adjoint of $M \circ dAss$ that may be called the *monoid differential algebra* of $\mathcal{M}$ and is denoted by ${}_{(R,\partial)}\mathbf{DiffAss}_1[\mathcal{M}]$. According to Theorem 18 it may also be described as

$$\begin{aligned} {}_{(R,\partial)}\mathbf{DiffAss}_1[\mathcal{M}] = {}_{(R,\partial)}\mathbf{DiffAss}_1[Mon(\mathcal{M})]/ \\ \langle j(xy) - j(x)j(y), j(e) - 1 \colon x, y \in Mon(\mathcal{M}) \rangle \end{aligned}$$

where e is the identity element of $\mathcal{M}$.

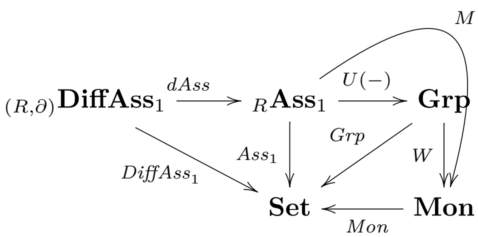
Proposition 39. *Any monoid $\mathcal{M}$ embeds into its monoid differential algebra ${}_{(R,\partial)}\mathbf{DiffAss}_1[\mathcal{M}]$. More precisely the insertion of generators $j \colon \mathcal{M} \to M(dAss({}_{(R,\partial)}\mathbf{DiffAss}_1[\mathcal{M}]))$, which, by definition, is a homomorphism of monoids, is one-to-one.*

Proof. It is well-known that the insertion of generators, say $i_{\mathcal{M}} \colon \mathcal{M} \to M(R[\mathcal{M}])$, seen as a set-theoretic map, is one-to-one. Now, define a (R,∂)-derivation on $R[\mathcal{M}]$ by $\mathbf{d}\left(\sum_{x\in\mathcal{M}} \alpha_x i_{\mathcal{M}}(x)\right) = \sum_{x\in\mathcal{M}} \partial(\alpha_x) i_{\mathcal{M}}(x)$ (where, in the sum, $\alpha_x \in R$ for each $x \in \mathcal{M}$, and there are only finitely many x such that $\alpha_x \neq 0$). This shows that $\mathcal{M}$ embeds (as a sub-monoid) into a (R,∂)-differential algebra by $i_{\mathcal{M}}$, hence it embeds into ${}_{(R,\partial)}\mathbf{DiffAss}_1[\mathcal{M}]$ using j because by the universal property satisfied by ${}_{(R,\partial)}\mathbf{DiffAss}_1[\mathcal{M}]$, there is a unique homomorphism $\hat{\phi} \colon {}_{(R,\partial)}\mathbf{DiffAss}_1[\mathcal{M}] \to (R[\mathcal{M}], \mathbf{d})$ of (R,∂)-differential algebras that extends $i_{\mathcal{M}}$, i.e., $\hat{\phi} \circ j = i_{\mathcal{M}}$, and given a set-theoretic retraction π of $i_{\mathcal{M}}$ (i.e., $\pi \circ i_{\mathcal{M}} = id$), one has $\pi \circ \hat{\phi} \circ j = \pi \circ i_{\mathcal{M}} = id$, hence j is also one-to-one. □

Remark 40. The construction, given in the proof of Proposition 39, of the (R,∂)-differential algebra $(R[\mathcal{M}], \mathbf{d})$ from the commutative differential ring (R,∂) extends to a functor.

Let $(\mathcal{A}, d)$ be any commutative (R, ∂)-differential algebra. The $\mathcal{A}$-algebra $\mathcal{A}[\mathcal{M}]$, namely the usual monoid $\mathcal{A}$-algebra of $\mathcal{M}$, is of course also a R-algebra (using the change of base ring along the unit map $R \to \mathcal{A}$, that sends the identity element of R to that of $\mathcal{A}$). Now, define a (R, ∂)-derivation on $\mathcal{A}[\mathcal{M}]$ by $\mathbf{d}\left(\sum_{x\in\mathcal{M}} \alpha_x i_{\mathcal{M}}(x)\right) = \sum_{x\in\mathcal{M}} d(\alpha_x) i_{\mathcal{M}}(x)$. Given a homomorphism of (R, ∂)-differential commutative algebras $\phi: (\mathcal{A}, d) \to (\mathcal{B}, e)$, one defines a homomorphism of (R, ∂)-differential algebras $\phi[\mathcal{M}]: (\mathcal{A}[\mathcal{M}], \mathbf{d}) \to (\mathcal{B}[\mathcal{M}], \mathbf{e})$ by setting $\phi[\mathcal{M}]\left(\sum_{x\in\mathcal{M}} \alpha_x i_{\mathcal{M}}(x)\right) = \sum_{x\in\mathcal{M}} \phi(\alpha_x) i_{\mathcal{M}}(x)$. It is now easy to check that $(-)[\mathcal{M}]: {}_{(R,\partial)}\mathbf{DiffAss}_{c,1} \to {}_{(R,\partial)}\mathbf{DiffAss}_1$ defines a functor.

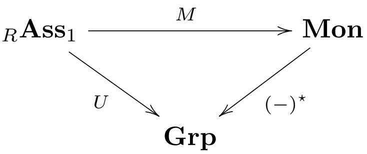
My next objective is to provide a similar construction for groups instead of monoids. One now considers the following diagram, whose leftmost and bottom right triangles and the (deformed) triangle, with vertices ${}_R\mathbf{Ass}_1$, **Mon**, **Set**, are commutative, where $U(-)$ is the usual functor that maps an algebra with a unit to its group of units (i.e., its invertible elements), M and W are the usual forgetful functors (M is as in the above Diagram (23), and W forgets the inverse map of a group).



(24)

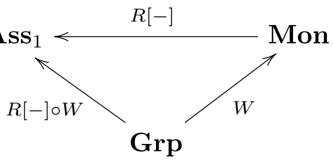
One observes that the composite $W \circ U$ is not the usual forgetful functor M from associative unital algebras to monoids but a *sub-functor* thereof (in the sense that, as sets, one has $Mon(W(U(\mathcal{A})) \subseteq Mon(M(\mathcal{A}))$ for every algebra $\mathcal{A}$ and for every algebra map $f: \mathcal{A} \to \mathcal{B}$, $incl \circ Mon(W(U(f))) = Mon(M(f)) \circ incl$, where $incl$ denotes an obvious canonical inclusion). Hence the (deformed) triangle with vertices ${}_R\mathbf{Ass}_1$, **Grp**, **Mon** is not commutative. Likewise the central triangle of the above diagram (the upper triangle in the square) is not commutative in general because $Grp \circ U$ is only a sub-functor of Ass_1.

The bottom right triangle falls into the scope of Theorem 18. The left adjoint of W is the well-known Grothendieck group completion (see Example 21.1). Even if it is outside of the scope of Theorem 18, the functor U is known to admit a left adjoint, namely the group algebra $R[\mathcal{G}]$, where $\mathcal{G}$ is a group, which is obtained from a left adjoint of the algebraic functor M (the monoid algebra functor). In details, $R[\mathcal{G}] = R[W(\mathcal{G})]$. Contrary to what it seems, it is a composition of left adjoints because W admitting a right adjoint, namely the functor $(-)^\star$ that sends a monoid to its group of invertible elements, is itself a left adjoint. In brief, one has the following commutative diagram of forgetful functors.



(25)

which induces an evident commutative diagram of left adjoints.



(26)

The left triangle of Diagram (24) falls into the construction of Section 4 and a description of a left adjoint of the functor *dAss* was already given in Example 26.2. Finally one obtains a left adjoint of $U(-) \circ dAss\colon {}_{(R,\partial)}\mathbf{DiffAss}_1 \to \mathbf{Grp}$ by composing left adjoints (see [21]): ${}_{(R,\partial)}\mathbf{DiffAss}_1[\mathcal{G}] = {}_{(R,\partial)}\mathbf{DiffAss}_1[R[\mathcal{G}]]$. Whence the differential algebra ${}_{(R,\partial)}\mathbf{DiffAss}_1[\mathcal{G}]$ may be called the *group differential algebra* of $\mathcal{G}$. Of course, as in the case of a monoid, a group $\mathcal{G}$ embeds into its group differential algebra as a sub-group of its group of units.

Remark 41. Since ${}_{(R,\partial)}\mathbf{DiffAss}_1[\mathcal{G}] = {}_{(R,\partial)}\mathbf{DiffAss}_1[R[W(\mathcal{G})]]$ one has the following presentation ${}_{(R,\partial)}\mathbf{DiffAss}_1[\mathcal{G}] = {}_{(R,\partial)}\mathbf{DiffAss}_1[Mon(W(\mathcal{G}))]/\langle j(xy) - j(x)j(y), j(e) - 1\colon x, y \in Mon(W(\mathcal{G}))\rangle$ where e is the identity element of $\mathcal{G}$. If $x \in \mathcal{G}$, then one denotes by $x^{(0)}$ its image in ${}_{(R,\partial)}\mathbf{DiffAss}_1[\mathcal{G}]$, and its nth derivative is denoted by $x^{(n)}$; in particular, $(x^{(0)})' = x^{(1)}$. Then, in ${}_{(R,\partial)}\mathbf{DiffAss}_1[\mathcal{G}]$, one recovers some of the usual laws for the derivation: $x^{(0)}(x^{-1})^{(0)} = (xx^{-1})^{(0)} = e^{(0)} = 1$, hence $0 = (x^{(0)}(x^{-1})^{(0)})' = x^{(1)}(x^{-1})^{(0)} + x^{(0)}(x^{-1})^{(1)}$ so that $(x^{-1})^{(1)} = -x^{(1)}(x^{-2})^{(0)}$, and more generally for $x, y \in \mathcal{G}$,

$$\begin{aligned}(x^{(0)}(y^{-1})^{(0)})' &= x^{(1)}(y^{-1})^{(0)} + x^{(0)}(y^{(-1)})' \\ &= x^{(1)}(y^{-1})^{(0)} - x^{(0)}y^{(1)}(y^{-2})^{(0)} \\ &= (x^{(1)}y^{(0)} - x^{(0)}y^{(1)})(y^{-2})^{(0)}. \end{aligned} \tag{27}$$

6. Two approaches for the universal enveloping differential algebra of a differential Lie algebra

In this section I consider the construction of a universal enveloping differential algebra of a differential Lie algebra. As explained in Section 1 there are actually (at least) two ways to treat the problem, and only the second one cannot be reached by usual constructions from non-differential algebras. I recall here the important fact that homomorphisms between differential (Lie) algebras are required to commute with the derivations.

6.1. The "lifted" universal enveloping algebra

Given a (R, ∂)-differential (associative and unital) algebra $(\mathcal{A}, d)$ (with $\mathcal{A} = ((A, +, 0, \cdot), *, 1)$), one has for $x, y \in A$,

$$\begin{aligned} d([x, y]) &= d(x * y - y * x) \\ &= d(x) * y + x * d(y) - d(y) * x - y * d(x) \\ &= [d(x), y] + [x, d(y)] \end{aligned} \tag{28}$$

so that d also is a (R, ∂)-derivation for the commutator bracket, and $(((A, +, 0, \cdot), [-, -]), d)$ turns to be a (R, ∂)-differential Lie algebra (see Example 26.1 for the definition of a (R, ∂)-differential Lie algebra).

Actually, this gives rise to a functor $dComm\colon {}_{(R,\partial)}\mathbf{DiffAss}_1 \to {}_{(R,\partial)}\mathbf{DiffLie}$ which makes commute the following diagram (of functors), where $Comm$ is the usual functor that consists in viewing an algebra as a Lie algebra under its *Comm*utator bracket; the other two functors, as usually, just forget the derivation.

$$\begin{array}{ccc} {}_{(R,\partial)}\mathbf{DiffAss}_1 & \xrightarrow{dComm} & {}_{(R,\partial)}\mathbf{DiffLie} \\ {\scriptstyle dAss}\downarrow & & \downarrow{\scriptstyle dLie} \\ {}_{R}\mathbf{Ass}_1 & \xrightarrow[Comm]{} & {}_{R}\mathbf{Lie} \end{array} \tag{29}$$

Because all the functors occurring in Diagram (29) are algebraic functors, each of them admits a left adjoint. A left adjoint of $Comm$ is the usual universal enveloping algebra functor U. Left adjoints of $dAss$ and $dLie$ have already been described, both in Example 26 (points (2) and (1)).

The task is now to provide an explicit construction for a left adjoint of $dComm$. So let $((g, [-, -]), d)$ be a (R, ∂)-differential Lie algebra. As in Subsection 5.3, one may consider the tensor differential algebra $(\mathsf{T}(g), \tilde{D})$ over the (R, ∂)-differential module (g, d). The (R, ∂)-derivation $\tilde{D}$ satisfies $\tilde{D}(j(x)j(y) - j(y)j(x) - j([x, y])) = j(d(x))j(y) + j(x)j(d(y)) - j(d(y))j(x) - j(y)j(d(x)) - j([d(x), y]) - j([x, d(y)])$ for every $x, y \in g$, where $j\colon g \to \mathsf{T}(g)$ is the canonical injection. Hence there is a unique R-linear map $\hat{D}\colon \mathsf{U}(g, [-, -]) \to \mathsf{U}(g, [-, -])$, which is easily seen to be a (R, ∂)-derivation, such that the following diagram commutes, where $\pi\colon \mathsf{T}(g) \to \mathsf{U}(g, [-, -]) = \mathsf{T}(g)/\langle j(x)j(y) - j(y)j(x) - j([x, y])\colon x, y \in g\rangle$ is the canonical epimorphism.

$$\begin{array}{ccc} \mathsf{T}(g) & \xrightarrow{\tilde{D}} & \mathsf{T}(g) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathsf{U}(g, [-, -]) & \xrightarrow[\hat{D}]{} & \mathsf{U}(g, [-, -]) \end{array} \tag{30}$$

Remark 42. The R-linear map $\pi \circ j: g \to \mathsf{U}(g, [-,-])$ is a homomorphism of differential Lie algebras from $((g, [-,-]), d)$ to $dComm(\mathsf{U}(g, [-,-]), \hat{D})$. This is clear because $\pi \circ j$ is a homomorphism of Lie algebras from $(g, [-,-])$ to $dComm(\mathsf{U}(g, [-,-]))$, and by its definition it commutes with the derivations.

Proposition 43. *The (R, ∂)-differential algebra $(\mathsf{U}(g, [-,-]), \hat{D})$ is the free (R, ∂)-differential algebra generated by the differential Lie algebra $((g, [-,-]), d)$. In details: for any (R, ∂)-differential (associative and unital) algebra $(\mathcal{A}, e)$ and for any homomorphism of differential Lie algebras $\phi: ((g, [-,-]), d) \to dComm(\mathcal{A}, e) = ((A, [-,-]), e)$ (where A is the underlying R-module of $\mathcal{A}$), there is a unique homomorphism of differential algebras $\hat{\phi}: (\mathsf{U}(g, [-,-]), \hat{D}) \to (\mathcal{A}, e)$ such that $\hat{\phi} \circ (\pi \circ j) = \phi$.*

Proof. ϕ is of course a homomorphism of Lie algebras from $(g, [-,-])$ to $(A, [-,-])$. Hence there is a unique algebra map $\hat{\phi}: \mathsf{U}(g, [-,-]) \to \mathcal{A}$ such that $\hat{\phi} \circ \pi \circ j = \phi$ (by the universal property of U). It remains to check that $\hat{\phi}$ commutes with the derivations. Let $\tilde{\phi}: (\mathsf{T}(g), \tilde{D}) \to (\mathcal{A}, e)$ be the unique homomorphism of (R, ∂)-differential algebras such that $\tilde{\phi} \circ j = \phi$ (by Proposition 36). By definition, $\tilde{\phi} \circ \tilde{D} = e \circ \bar{\phi}$. Then, one has

$$\hat{\phi} \circ \hat{D} \circ \pi = \hat{\phi} \circ \pi \circ \tilde{D} = \tilde{\phi} \circ \tilde{D} = e \circ \tilde{\phi} = e \circ \hat{\phi} \circ \pi.$$

Therefore $\hat{\phi} \circ \hat{D} = e \circ \hat{\phi}$ (since π is onto). $\square$

The differential algebra ${}_{(R,\partial)}\mathbf{DiffAss}_1[(g, [-,-]), d] = (\mathsf{U}(g, [-,-]), \hat{D})$ is referred to as the *universal enveloping differential algebra* of the differential Lie algebra $((g, [-,-]), d)$. The following diagram commutes, which shows that the universal enveloping differential algebra is obtained by lifting to the differential algebra setting the usual universal enveloping algebra construction.

$$\begin{array}{ccc} {}_{(R,\partial)}\mathbf{DiffLie} & \xrightarrow{{}_{(R,\partial)}\mathbf{DiffAss}_1[-]} & {}_{(R,\partial)}\mathbf{DiffAss}_1 \\ {\scriptstyle dLie}\downarrow & & \downarrow{\scriptstyle dAss} \\ {}_{R}\mathbf{Lie} & \xrightarrow[\mathsf{U}]{} & {}_{R}\mathbf{Ass}_1 \end{array} \tag{31}$$

It thus readily follows that Poincaré–Birkhoff–Witt Theorem holds unchanged for differential Lie algebras.

Theorem 44 *(Poincaré–Birkhoff–Witt Theorem for differential Lie algebras). A differential Lie algebra embeds, through $\pi \circ j$, as a differential sub-Lie algebra of its universal enveloping differential algebra (under its commutator bracket) if, and only if, its underlying Lie algebra embeds into its universal enveloping algebra (under the commutator bracket), again by $\pi \circ j$.*

Example 45. For instance, if g is a free R-module, then $(g,[-,-])$ is a sub-Lie algebra of the Lie algebra $(\mathsf{U}(g,[-,-]),[-,-]) = \mathit{Comm}(\mathsf{U}(g,[-,-]))$, hence the map $\pi\circ j\colon ((g,[-,-]),d)\to d\mathit{Comm}(\mathsf{U}(g,[-,-]),\hat{D})$ is one-to-one.

6.2. Wronskian enveloping algebra

One observes that whenever $(\mathcal{A},d)$ is a commutative differential algebra, then $d\mathit{Comm}(\mathcal{A},d)$ turns to be a commutative differential Lie algebra, i.e., a Lie algebra with a zero bracket, hence just a differential module. Moreover, given a commutative differential Lie algebra $((g,0),d)$, from Subsection 6.1 it follows that ${}_{(R,\partial)}\mathbf{DiffAss}_1[(g,0),d] = (\mathsf{U}(g,0),\hat{D}) = (\mathsf{S}(g),\hat{D})$ (because the universal enveloping algebra of any commutative Lie algebra is the symmetric algebra of its underlying module; see [6]). It is also clear that $\hat{D}$ actually corresponds to the derivation $\tilde{D}$ on $\mathsf{S}(g)$ as defined in Subsection 5.3. Therefore, the universal enveloping differential algebra of a commutative differential Lie algebra is its symmetric differential algebra.

It thus seems uninteresting to consider differential commutative algebras from the point of view of their relations with differential Lie algebras. But I argue below that it is not the case at all, and it is even the complete opposite! Indeed if one restricts our attention to *commutative* differential algebras only, then there is another "forgetful" functor, which is algebraic, from ${}_{(R,0)}\mathbf{DiffAss}_{c,1}$ to ${}_{(R,0)}\mathbf{DiffLie}$, that differs from $d\mathit{Comm}$, and that cannot be obtained as a lift of a functor from ${}_R\mathbf{Ass}_{c,1}$ to ${}_R\mathbf{Lie}$.

Hence, let $(((A,+,0,\cdot),*,1),d)$ be a commutative (unital and associative) $(R,0)$-differential algebra. Let us recall the definition of the *Wronskian bracket* (of course, the name comes from the Wronskian operation known in the case of differentiable functions; see for instance [5]) as given in Section 1. Let $x,y\in A$. Then, $W(x,y) = x*d(y)-d(x)*y$. Of course, this map is R-bilinear, and alternating, i.e., $W(x,x) = x*d(x)-d(x)*x = x*d(x)-x*d(x) = 0$. Moreover it satisfies the Jacobi identity (commutativity of $*$ is essential to see this). Therefore, $((A,+,0,\cdot),W)$ is a Lie R-algebra.

Remark 46. If $(((A,+,0,\cdot),*,1),d)$ is a commutative (unital and associative) (R,∂)-differential algebra, then its Wronskian bracket is only bi-additive (i.e., $\mathbb{Z}$-bilinear) and, in general, not R-bilinear. Indeed, it satisfies the following: $W(x,\alpha\cdot y) = \partial(\alpha)\cdot x*y+\alpha\cdot W(x,y)$ for each $x,y\in A$ and $\alpha\in R$ (a relation that makes it a "Lie pseudo-algebra"; see, e.g., [15]). This explains why the base ring R is equipped with the zero derivation.

Moreover, for $x,y\in A$,

$$\begin{aligned} d(W(x,y)) &= d(x*d(y)-d(x)*y)\\ &= d(x)*d(y)+x*d^2(y)-d^2(x)*y-d(x)*d(y)\\ &= x*d^2(y)-d^2(x)*y, \end{aligned} \tag{32}$$

while

$$W(d(x), y) + W(x, d(y)) = d(x) * d(y) - d^2(x) * y + x * d^2(y) - d(x) * d(y).$$

Hence d is also a R-derivation for the Wronskian bracket, so that $(((A, +, 0, \cdot), W), d)$ turns to be a $(R, 0)$-differential Lie algebra. The correspondence $(((A, +, 0, \cdot), *, 1), d) \mapsto (((A, +, 0, \cdot), W), d)$ is the object map of a functor $dWron\colon {}_{(R,0)}\mathbf{DiffAss}_{c,1} \to {}_{(R,0)}\mathbf{DiffLie}$ (because for each homomorphism ϕ of differential algebras, $\phi(W(x, y)) = \phi(x * d(y) - d(x) * y) = \phi(x) * d(\phi(y)) - d(\phi(x)) * \phi(y) = W(\phi(x), \phi(y)))$ which I call the *Wronskian functor*.

The Wronskian functor is an algebraic functor, hence it admits a left adjoint, say $\mathcal{W}$. Let $(\mathfrak{g}, d)$ be a $(R, 0)$-differential Lie algebra (with underlying set g). According to Theorem 18, it holds that $\mathcal{W}(\mathfrak{g}, d) = {}_{(R,0)}\mathbf{DiffAss}_{c,1}[g]/\langle j_g(x+y) - j_g(x) - j_g(y), j_g(\alpha \cdot g) - \alpha j_g(x), j_g(0), j_g([x, y]) - j_g(x)(j_g(y))' + (j_g(x))' j_g(y), j_g(d(x)) - (j_g(x))' \colon x, y \in g, \alpha \in R\rangle$. But ${}_{(R,0)}\mathbf{DiffAss}_{c,1}[g]$ is the algebra $R\{g\}$ of differential polynomials in the commutative variables in g (with the zero derivation on R). It is more usual to denote by $x^{(0)}$ the member $j_g(x)$ of $R\{g\}$, $x \in g$, and the derivation then is $(x^{(0)})' = x^{(1)}$. Thus, $\mathcal{W}(\mathfrak{g}, d)$ is the quotient algebra of $R\{g\}$ by the differential ideal I_g generated by

$$\begin{aligned}(x+y)^{(0)} - x^{(0)} - y^{(0)},\ (\alpha \cdot x)^{(0)} - \alpha x^{(0)},\ 0^{(0)},\\ [x, y]^{(0)} - x^{(0)} y^{(1)} + x^{(1)} y^{(0)},\ (d(x))^{(0)} - x^{(1)},\end{aligned}$$

$x, y \in g$, $\alpha \in R$.

One calls *universal Wronskian envelope* of $(\mathfrak{g}, d)$ the commutative differential algebra $\mathcal{W}(\mathfrak{g}, d)$ in order not to confuse with the universal enveloping differential algebra from Subsection 6.1.

Remark 47. Let us assume that $(\mathfrak{g}, 0)$ is a $(R, 0)$-differential Lie algebra with a zero derivation (hence essentially just a usual Lie algebra). Then, because $0^{(0)} = (d(x))^{(0)}$ is equivalent both to 0 and to $x^{(1)}$ mod I_g, it follows that the derivation on $R\{g\}/I_g$ is also the zero derivation (since $x^{(n)}$ generates $R\{g\}$, and $x^{(n+1)} \cong 0$ mod I_g for each n, $x \in g$). Despite an apparent, but misleading, similarity, this construction is quite different from the construction of the *Wronskian envelope* of $\mathfrak{g}$ studied in [24]. The latter corresponds to the free commutative $(R, 0)$-differential algebra ${}_{(R,0)}\mathbf{DiffAss}_{c,1}[\mathfrak{g}]$ generated by the Lie algebra $\mathfrak{g}$, which by Theorem 18 is given as the quotient differential algebra ${}_{(R,0)}\mathbf{DiffAss}_{c,1}[g]/\langle j_g(x + y) - j_g(x) - j_g(y), j_g(\alpha \cdot x) - \alpha j_g(x), j_g(0), j_g([x, y]) - j_g(x)(j_g(y))' + (j_g(x))' j_g(y) \colon x, y \in g, \alpha \in R\rangle$, or also, $R\{g\}/\langle (x + y)^{(0)} - x^{(0)} - y^{(0)}, (\alpha \cdot x)^{(0)} - \alpha x^{(0)}, 0^{(0)}, [x, y]^{(0)} - x^{(0)} y^{(1)} + x^{(1)} y^{(0)} \colon x, y \in g, \alpha \in R\rangle$, while the former is $\mathcal{W}(J(\mathfrak{g}))$ (recall from Subsection 5.1 that J is the full embedding of ${}_R\mathbf{Lie}$ into ${}_{(R,0)}\mathbf{DiffLie}$).

There is another possible description for $\mathcal{W}$. Let $(\mathfrak{g}, d)$ be a $(R, 0)$-differential Lie algebra (with underlying R-module g). Let us consider the symmetric differential algebra $(\mathsf{S}(g), \tilde{D})$ of the $(R, 0)$-differential module (g, d) given as in Subsection 5.3. Let us consider the (algebraic) ideal I generated by $h(x)h(d(y)) - h(d(x))h(y) - h([x, y])$, $x, y \in g$, where $h = \pi \circ j: g \to \mathsf{S}(g)$ is the canonical embedding. By definition of $\tilde{D}$, one has $\tilde{D}(h(x)) = h(d(x))$ for each $x \in g$, so that I is the ideal generated by $W(h(x), h(y)) - h([x, y])$, $x, y \in g$, with W being the Wronskian bracket of $(\mathsf{S}(g), \tilde{D})$.

One observes that $\tilde{D}(I) \subseteq I$. Indeed,

$$\begin{aligned} \tilde{D}(W(h(x), h(y)) &= W(\tilde{D}(h(x)), h(y)) + W(h(x), \tilde{D}(h(y))) \\ &= W(h(d(x)), h(y)) + W(h(x), h(d(y))) \end{aligned} \tag{33}$$

while

$$\begin{aligned} \tilde{D}(h([x, y])) &= h(d([x, y])) \\ &= h([d(x), y] + [x, d(y)]) \\ &\quad \text{(since } d \text{ is a derivation for the Lie bracket)} \\ &= h([d(x), y]) + h([x, d(y)]). \end{aligned} \tag{34}$$

Hence, I is already a differential ideal of $(\mathsf{S}(g), \tilde{D})$.

I claim that $(\mathsf{S}(g)/I, \bar{D})$ is the universal Wronskian envelope of $(\mathfrak{g}, d)$, where $\bar{D}$ is the quotient derivation given by $\bar{D}(x + I) = \tilde{D}(x) + I$. To see this, let $(\mathcal{A}, e)$ (with underlying R-module A) be any commutative (associative and unital) $(R, 0)$-differential algebra, and let $\phi: (\mathfrak{g}, d) \to dWron(\mathcal{A}, e) = ((A, W), e)$ be a homomorphism of $(R, 0)$-differential Lie algebras. Then, there is a unique algebra map $\hat{\phi}: \mathsf{S}(g) \to \mathcal{A}$ (where g is the underlying module of $\mathfrak{g}$) such that $\hat{\phi} \circ h = \phi$. The map $\hat{\phi}$ commutes with the derivations (because $\hat{\phi}(\tilde{D}(h(x))) = \hat{\phi}(h(d(x))) = \phi(d(x)) = e(\phi(x)) = e(\hat{\phi}(h(x)))$, and the $h(x)$'s generate $\mathsf{S}(g)$ when x ranges over g). Moreover it satisfies

$$\begin{aligned} &\hat{\phi}(h(x)h(d(y)) - h(d(x))h(y) - h([x, y])) \\ &\quad = \hat{\phi}(h(x))\hat{\phi}(h(d(y))) - \hat{\phi}(h(d(x)))\hat{\phi}(h(y)) - \hat{\phi}(h([x, y])) \\ &\quad = \phi(x)e(\phi(y)) - e(\phi(x))\phi(y) - W(\phi(x), \phi(y)) \\ &\quad = 0 \end{aligned} \tag{35}$$

for every $x, y \in g$. Hence it factors through I and provides a unique homomorphism of commutative differential algebras $\tilde{\phi}: (\mathsf{S}(g)/I, \bar{D}) \to (\mathcal{A}, e)$ such that $\tilde{\phi}(h(x) + I) = \phi(x)$ for each $x \in g$.

Remark 48. The R-linear map $\bar{h}: g \to \mathsf{S}(g)/I$ given by $\bar{h}(x) = h(x) + I$, $x \in g$, is a homomorphism of differential Lie algebras from $(\mathfrak{g}, d)$ to $((\mathsf{S}(g)/I, W), \bar{D})$. Indeed it of course commutes with the derivations by definition, and $\bar{h}([x, y]) = W(\bar{h}(x), \bar{h}(y))$ by the definition of I.

Therefore the following result is proved.

Proposition 49. *Using the same notations as above,* $(\mathsf{S}(g)/I, \bar{D})$ *is the universal Wronskian envelope of* $(\mathfrak{g}, d)$.

The following example shows that the question of the embedding conditions of a differential Lie algebra into its universal Wronskian envelope is much more difficult than in the non-differential case.

Example 50. The construction of $\mathcal{W}(\mathfrak{g}, d)$ given in Proposition 49 is useful to make the following observation. Let $(g, [-,-])$ be a Lie R-algebra. The derivation $\tilde{D}$ on $\mathsf{S}(g)$ that extends the zero derivation on $(g, [-,-])$ is also just the zero derivation (see also Remark 47). Hence the ideal I is here equal to the (algebraic) ideal of $\mathsf{S}(g)$ generated by $[h(x), h(y)]$, $x, y \in g$. Hence $\mathcal{W}((g, [-,-]), 0) = (\mathcal{S}(g)/I, 0)$. If $(g, [-,-])$ is not a commutative Lie algebra (i.e., when $[-,-]$ is not identically zero), $((g, [-,-]), 0)$ does not embed into its universal Wronskian envelope (because $\bar{h}([x, y]) = 0$ even if $[x, y] \neq 0$ for some $x, y \in g$), even if g is free as a R-module, and even if R is a field! Of course if $(g, [-,-])$ is commutative, then I is reduced to (0), and in this case $\bar{h} = h$ is an embedding into $\mathcal{W}((g, 0), 0) = (\mathsf{S}(g), 0)$.

A positive example of an embedding is given below.

Example 51. Let $\mathbb{K}$ be a field of characteristic zero. The Lie algebra $\mathfrak{sl}_2(\mathbb{K})$ embeds into the algebra of vector fields of $\mathbb{K}[x]$ by the identification of the elements of its Chevalley basis $e = -1$, $h = -2x$, $f = x^2$ (the familiar commutation rules are satisfied: $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$). This defines a $(\mathbb{K}, 0)$-differential Lie algebra when it is equipped with the usual derivation $\frac{d}{dx}$ of polynomials. More precisely, let $d: \mathfrak{sl}_2(\mathbb{K}) \to \mathfrak{sl}_2(\mathbb{K})$ be given on generators by $d(e) = 0$, $d(h) = 2e$ and $d(f) = -h$ (hence it corresponds to $\frac{d}{dx}$). One easily checks that $d([u, v]) = [d(u), v] + [u, d(v)]$ for all generators u, v of $\mathfrak{sl}_2(\mathbb{K})$, hence $(\mathfrak{sl}_2(\mathbb{K}), d)$ is a differential Lie algebra. It embeds as a differential sub-Lie algebra into $(\mathbb{K}[x], \frac{d}{dx})$ with the Wronskian bracket, therefore it embeds also into its universal Wronskian envelope.

7. Varieties of Rota–Baxter algebras

In this last section I want to demonstrate usefulness of the tools developed in Section 4 to a setting different from that of differential algebra, namely to the case of Rota–Baxter algebras [28], again focusing on the relation between Rota–Baxter associative algebras and their Lie counterpart.

Let, as in Subsection 5.1, ${}_R\mathbf{V}$ be a variety of (not necessarily associative nor unital) R-algebras, i.e., a sub-variety of either ${}_R\mathbf{Alg}$ or ${}_R\mathbf{Alg_1}$, and let $\mathcal{A} = (A, *)$ (or $(A, *, 1)$ in case of unital algebras) be a member of ${}_R\mathbf{V}$ (hence A is a R-module). By a *Rota–Baxter* *operator*[1] *on* $\mathcal{A}$ is meant a R-linear endomorphism $B\colon A \to A$ which satisfies the so-called *Rota–Baxter identity*

$$B(x) * B(y) = B(B(x) * y + x * B(y))$$

$x, y \in A$. A pair $(\mathcal{A}, B)$ is called a *Rota–Baxter algebra* in ${}_R\mathbf{V}$, and the algebra $\mathcal{A}$ is referred to as its *underlying algebra*.

Let ${}_R\mathbf{RBV}$ be the variety of all *Rota–Baxter algebras* in ${}_R\mathbf{V}$, the homomorphisms of which are given as homomorphisms between the underlying R-algebras that commute with the Rota–Baxter operators.

Examples 52.

1. Let ${}_R\mathbf{V}$ be the variety of all R-modules (i.e., algebras with a zero multiplication). Then, a Rota–Baxter operator is just a R-linear endomorphism (because in this case the Rota–Baxter identity degenerates to $B(0) = 0$). Hence ${}_R\mathbf{RBMod} = {}_{(R,0)}\mathbf{DiffMod}$.
2. Let ${}_R\mathbf{V} \in \{{}_R\mathbf{Ass}, {}_R\mathbf{Ass}_1, {}_R\mathbf{Ass}_c, {}_R\mathbf{Ass}_{c,1}\}$. With ${}_R\mathbf{RBV}$ is thus obtained the varieties of associative Rota–Baxter algebras, unital or not, commutative or not.
3. Let ${}_R\mathbf{V} = {}_R\mathbf{Lie}$. Then, ${}_R\mathbf{RBLie}$ is the variety of Rota–Baxter Lie algebras. The Rota–Baxter identity then reads as follows $[B(x), B(y)] = B([B(x), y] + [x, B(y)])$.

Of course, ${}_R\mathbf{RBV}$ forms an equational variety of algebras, hence it admits a free object ${}_R\mathbf{RBV}[X]$ on any set X. By a two-sided *Rota–Baxter ideal* I of a Rota–Baxter algebra $(\mathcal{A}, B)$ is meant a two-sided ideal of $\mathcal{A}$, closed under B, i.e., $B(I) \subseteq I$. Hence $\tilde{B}(x) = B(x) + I$ defines a Rota–Baxter operator on $\mathcal{A}/I$. For any set $E \subseteq \mathcal{A}$, there is a least Rota–Baxter ideal on $\mathcal{A}$ that contains E, called the *Rota–Baxter ideal generated by* E. It is nothing else than the usual ideal generated by $\{B^n(x)\colon n \in \mathbb{N},\ x \in E\}$. Finally, congruences and two-sided Rota–Baxter ideals are in a one–one correspondence.

One observes that the zero map is a Rota–Baxter operator for any algebra in any variety of algebras (since the zero map satisfies the trivial Rota–Baxter identity $0 = 0$). Moreover, because any homomorphism of algebras maps 0 to 0, this gives rise to a functor $J\colon {}_R\mathbf{V} \to {}_R\mathbf{RBV}$ that maps $\mathcal{A}$ to $(\mathcal{A}, 0)$. It is a full and faithful functor that enables to identify ${}_R\mathbf{V}$ as a (full) sub-category of ${}_R\mathbf{RBV}$ (namely the full sub-category of ${}_R\mathbf{RBV}$ spanned by those Rota–Baxter algebras of the form $(\mathcal{A}, 0)$).

Proposition 53. ${}_R\mathbf{V}$ *is a reflective sub-category of* ${}_R\mathbf{RBV}$*. More precisely, J admits a left adjoint. In details: let $(\mathcal{A}, B)$ be a member of* ${}_R\mathbf{RBV}$*. There exists an algebra $\mathcal{A}^*$ in* ${}_R\mathbf{V}$ *and a homomorphism of Rota–Baxter algebras $\pi\colon (\mathcal{A}, B) \to J(\mathcal{A}^*) = (\mathcal{A}^*, 0)$ such that for each algebra $\mathcal{B}$ in* ${}_R\mathbf{V}$ *and each homomorphism of Rota–Baxter algebras*

[1] I restrict the study to Rota–Baxter operators of *weight zero*.

$\phi\colon (\mathcal{A}, B) \to (\mathcal{B}, 0) = J(\mathcal{B})$, *there is a unique homomorphism of algebras* $\hat{\phi}\colon \mathcal{A}^* \to \mathcal{B}$ *that satisfies* $\hat{\phi} \circ \pi = \phi$.

Proof. Let I_B be the two-sided ideal of $\mathcal{A}$ generated by $im(B)$. Then, $\mathcal{A}/I_B$ is a member of ${}_R\mathbf{V}$, and let $\pi\colon \mathcal{A} \to \mathcal{A}/I_B$ be the canonical epimorphism. Given an algebra $\mathcal{B}$ in ${}_R\mathbf{V}$ and a homomorphism $\phi\colon (\mathcal{A}, B) \to J(\mathcal{B}) = (\mathcal{B}, 0)$ of Rota–Baxter algebras, since $\phi \circ B = 0$, ϕ factors uniquely through π and provides a homomorphism $\hat{\phi}\colon \mathcal{A}/I_B \to \mathcal{B}$ such that $\hat{\phi} \circ \pi = \phi$. □

In an opposite way the obvious forgetful functor ${}_{rb}V\colon {}_R\mathbf{RBV} \to {}_R\mathbf{V}$ also admits a left adjoint (because it is an algebraic functor). Its construction follows at once by Theorem 18: let $\mathcal{A} = ((A, +, 0, \cdot), *)$ be an algebra in the variety ${}_R\mathbf{V}$, then ${}_R\mathbf{RBV}[\mathcal{A}] = {}_R\mathbf{RBV}[A]/I_{\mathcal{A}}$, where $I_{\mathcal{A}}$ is the (two-sided) Rota–Baxter ideal generated by

$$j_A(x + y) - j_A(x) - j_A(y), j_A(\alpha \cdot x) - \alpha j_A(x), j_A(0), j_A(x * y) - j_A(x) j_A(y)$$

(and also by $j_A(1) - 1$ when ${}_R\mathbf{V}$ is a variety of unital algebras), and $\pi \circ j_A\colon \mathcal{A} \to {}_{rb}V({}_R\mathbf{RBV}[A]/I_A)$ is the insertion of generators.

Let $(((A, +, 0, \cdot), *), B)$ be an associative Rota–Baxter algebra, i.e., a member of the variety ${}_R\mathbf{RBAss}$. It is possible to define its *double product* (see [12]) as

$$x *_B y := B(x) * y + x * B(y),$$

$x, y \in A$. It is left to the reader to check that this bilinear product is associative, and thus $((A, +, 0, \cdot), *_B)$ is once again an associative algebra. Moreover $B \in {}_R\mathbf{Ass}(Dbl(\mathcal{A}), \mathcal{A})$, i.e., B is a homomorphism of algebras from the algebra $((A, +, 0, \cdot), *_B)$ to the original algebra $((A, +, 0, \cdot), *)$. Finally B is also a Rota–Baxter operator for the algebra $(((A, +, 0, \cdot), *_B)$. Indeed,

$$B(x) *_B B(y) = B^2(x) * B(y) + B(x) * B^2(y) = B(B(x) *_B y + x *_B B(y)).$$

So B is a homomorphism of Rota–Baxter algebras $B\colon (((A, +, 0, \cdot), *_B), B) \to (((A, +, 0, \cdot), *,)B)$.

Given a homomorphism $\phi\colon (\mathcal{A}, B) \to (\mathcal{B}, C)$ of Rota–Baxter (associative) algebras, one has

$$\begin{aligned} \phi(x *_B y) &= \phi(B(x) * y + x * B(y)) \\ &= \phi(B(x)) * \phi(y) + \phi(x) * \phi(B(y)) \\ &= C(\phi(x)) * \phi(y) + \phi(x) * C(\phi(y)) \\ &= \phi(x) *_C \phi(y) \end{aligned} \tag{36}$$

for all $x, y \in A$. It thus follows that one gets a functor "double" $Dbl\colon {}_R\mathbf{RBAss} \to {}_R\mathbf{RBAss}$.

Remark 54. In a similar way one defines a functor *Dbl*: ${}_R\mathbf{RBAss}_1 \to {}_R\mathbf{RBAss}$, because an algebra with a double product has no obvious identity element.

A similar phenomenon occurs in the case of Rota–Baxter Lie algebras: given a Rota–Baxter Lie algebra $((g, [-,-]), B)$ one defines its *double bracket* as the R-bilinear map

$$[x, y]_B = [B(x), y] + [x, B(y)],$$

$x, y \in g$.

The double bracket is of course alternating (since $[-,-]$ is so). It satisfies the Jacobi identity. Indeed, $[-,-]$ itself satisfies Jacobi identity and the following equalities hold.

$$\begin{aligned}
[x,[y,z]_B]_B &= [B(x),[y,z]_B] + [x, B([y,z]_B)] \\
&= [B(x),[B(y),z] + [y,B(z)]] + [x, B([B(y),z] + [y,B(z)])] \\
&= [B(x),[B(y),z]] + [B(x),[y,B(z)]] + [x,[B(y),B(z)]], \\
[[x,y]_B,z]_B &= [B([x,y]_B),z] + [[x,y]_B,B(z)] \\
&= [B([B(x),y] + [x,B(y)]),z] + [[B(x),y] + [x,B(y)],B(z)] \\
&= [[B(x),B(y)],z] + [[B(x),y],B(z)] + [[x,B(y)],B(z)], \\
[y,[x,z]_B]_B &= [B(y),[x,z]_B] + [y,B([x,z]_B)] \\
&= [B(y),[B(x),z] + [x,B(z)]] + [y,B([B(x),z] + [x,B(z)])] \\
&= [B(y),[B(x),z]] + [B(y),[x,B(z)]] + [y,[B(x),B(z)]].
\end{aligned}$$

Hence $(g, [-,-]_B)$ is again a Lie algebra.

Moreover, $B([x,y]_B) = B([B(x),y] + [x,B(y)]) = [B(x),B(y)]$, hence B is a homomorphism of Lie algebras from $(g,[-,-]_B)$ to $(g,[-,-])$. It follows that

$$[B(x),B(y)]_B = [B^2(x),B(y)] + [B(x),B^2(y)] = B([B(x),y] + [x,B(y)]) = B([x,y]_B)$$

hence $(g, [-,-]_B, B)$ is a Rota–Baxter Lie algebra.

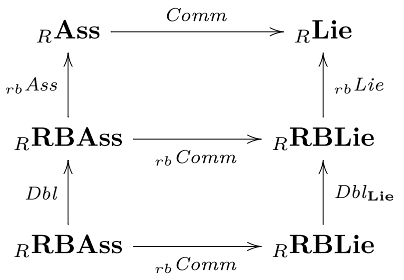
Quite obviously this provides a functor Dbl_{Lie}: ${}_R\mathbf{RBLie} \to {}_R\mathbf{RBLie}$.

There is also a functorial way to relate Rota–Baxter (associative) algebras to Rota–Baxter Lie algebras. Let $(((A,+,0,\cdot),*),B)$ be a Rota–Baxter associative algebra. Then, $(((A,+,0,\cdot),[-,-]),B)$ is a Rota–Baxter Lie algebra (where $[-,-]$ denotes the commutator bracket). Indeed,

$$\begin{aligned}
[B(x),B(y)] &= B(x)*B(y) - B(y)*B(x) \\
&= B(B(x)*y + x*B(y) - B(y)*x - y*B(x)) \\
&= B(B(x)*y - y*B(x) + x*B(y) - B(y)*x) \\
&= B([B(x),y] + [x,B(y)]) \qquad (37)
\end{aligned}$$

$x, y \in A$.

Therefore, one gets a functor ${}_{rb}Comm\colon {}_{R}\mathbf{RBAss} \to {}_{R}\mathbf{RBLie}$, and the following diagram commutes.



Each of these functors is algebraic, hence admits a left adjoint. In particular, one can form the *universal enveloping Rota–Baxter algebra* on a Rota–Baxter Lie algebra.

8. Conclusion

Under the lifted version (Subsection 6.1), the question of the embeddability of a differential Lie algebra into its universal enveloping differential algebra was shown rather simple since it only depends on the same question at the non-differential level, which is pretty well controlled by the famous Poincaré–Birkhoff–Witt Theorem.

On the contrary the same question for the Wronskian bracket version seems to be quite harder (see Example 50). It also seems to be connected to the existence of a (faithful) realization of a Lie algebra as a Lie algebra of vector fields over a one-dimensional smooth variety. For instance, given two polynomial vector fields $P(x)\frac{d}{dx}$ and $Q(x)\frac{d}{dx}$, their commutator bracket $[P(x)\frac{d}{dx}, Q(x)\frac{d}{dx}]$ (as operators on the polynomial algebra $R[x]$) is equal to $W(P(x), Q(x))\frac{d}{dx}$. Nevertheless Lie algebras of vector fields satisfy some non-trivial identities (i.e., identities which are not satisfied by every Lie algebras) as explained in the unpublished note [2]. This makes this problem more intricate since specific to a particular kind of Lie algebras.

Another possible way to deal with this problem would be to use Gröbner bases for differential algebras [22] that should provide a basis for the quotient algebra $\mathcal{W}(\mathfrak{g}, d)$ at least when the base ring is a field.

Finally, as suggested by a referee, Herz's work [15] on several associative envelopes of Lie pseudo-algebras over a field should be investigated in order to obtain results about the embedding conditions of a (differential or not) Lie algebra into its Wronskian envelope.

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the Wronskian bracket in the literature on differential algebraic Lie algebras that might be useful to get a better understanding of the relations between a Lie algebra and its Wronskian envelope.

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